

Note

An Asymptotic Formula for the Number of Classes of Sets of n Indistinguishable Elements

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Communicated by S. M. Ulam

Received August 14, 1974

Letting X be a finite set and P the power set operation on a set, an approximation for the number of members of $P^d(X)$, for $d > 2$, is derived, where two members are considered identical if some permutation on X can transform one into the other. The approximation is asymptotic to the expected lower bound $|P^d(X)|/|X|!$

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This note addresses the following problem posed by S. M. Ulam. Find a concise expression for the number of classes of sets of n indistinguishable elements. Here, a class of sets is a set whose members are subsets of a given finite set of n elements, and referring to the elements as indistinguishable means that two classes are identical if some permutation on the elements transforms one class into the other. Harary [2] gives some formulas which, when applied together with Pólya's well-known enumeration theorem [3], result in an exact expression for the number of classes. Unfortunately, this expression involves a complicated sum over either the permutations on n or the partitions on n .

The problem may be generalized in the obvious way to the hierarchy of classes of classes of sets and so forth. The following results apply not only to classes of sets but to this hierarchy as well, giving simple asymptotic estimates.

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Let X be a finite set with n elements and S_X the symmetric group on X . We make the following recursive definitions. For any set A ,

$P(A) = \{B \mid B \subseteq A\}$, $P^0(A) = A$, and $P^{d+1}(A) = P(P^d(A))$. $p(n, 0) = n$ and $p(n, d + 1) = 2^{p(n, d)}$. Thus, $|P^d(X)| = p(n, d)$.

Let $\alpha \in S_X$. We define recursively the action of α on $P^d(X)$ for $d = 1, 2, \dots$, putting $\alpha(A) = \{\alpha(a); a \in A\}$ and we define an equivalence relation \equiv on $P^d(X)$ putting $A \equiv B$ if there exists some $\alpha \in S_X$ such that $\alpha(A) = \alpha(B)$. Let $N(n, d)$ be the number of equivalence classes of \equiv . Thus

$$N(n, 0) = 1 \quad \text{and} \quad N(n, 1) = n + 1.$$

Given $\alpha \in S_X$, we may describe the cycle structure of α as a permutation on $P^d(X)$ by a $p(n, d)$ -dimensional vector $(C_1(d, \alpha), \dots, C_{p(n, d)}(d, \alpha))$, where $C_i(d, \alpha)$ is the number of cycles of α on $P^d(X)$ of length i . In what follows, $C_1(d, \alpha)$, the number of unary cycles, i.e., fixed points of α on $P^d(X)$, is of particular importance.

LEMMA. $C_1(d + 1, \alpha) = \prod_{i=1}^{p(n, d)} 2^{C_i(d, \alpha)}$.

Proof. Let $A \in P^{d+1}(X)$ and assume $\alpha(A) = A$. Then for any cycle of α on $P^d(X)$, either all members of the cycle are members of A or no members of the cycle are members of A . This implies the Lemma.

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We put

$$q(n, 0) = n - 2, \quad r(n, 0) = n - 3,$$

and

$$q(n, d + 1) = 2^{(1/2)(p(n, d) + q(n, d))}, \quad r(n, d + 1) = 2^{(1/2)(p(n, d) + r(n, d))}.$$

THEOREM. For $d \geq 2$,

$$\frac{p(n, d)}{n!} + \frac{q(n, d)}{2(n - 2)!} < N(n, d) < \frac{p(n, d)}{n!} + \frac{q(n, d)}{2(n - 2)!} + \frac{\left(n! - \binom{n}{2} - 1\right) r(n, d)}{n!}.$$

Proof. By Burnside's Lemma (actually due to Frobenius [1]) we have

$$N(n, d) = \frac{1}{n!} \sum_{\alpha \in S_X} C_1(d, \alpha).$$

This sum will be separated into three parts by classifying members of

S_X according to their cycle structure. Let e be the identify permutation on X . Let J consist of those permutations of X which have one 2-cycle and $n - 2$ unary cycles, and let $K = S_X - (\{e\} \cup J)$. Thus,

$$N(n, d) = \frac{1}{n!} (C_1(d, e) + \sum_{\alpha \in J} C_1(d, \alpha) + \sum_{\alpha \in K} C_1(d, \alpha)).$$

Clearly $C_1(d, e) = p(n, d)$. If we show that for $\alpha \in J$, $C_1(d, \alpha) = q(n, d)$ and for $\alpha \in K$, $C_1(d, \alpha) \leq r(n, d)$, the theorem will be proved.

Let $\alpha \in J$. We use induction on d to prove $C_1(d, \alpha) = q(n, d)$. This is clear for $d = 0$. Now let us assume the result is true for d and prove it for $d + 1$.

Since $C_i(d, \alpha) = 0$ for $i > 2$, $C_1(d, \alpha) + 2C_2(d, \alpha) = p(n, d)$. Therefore $\sum C_i(d, \alpha) = \frac{1}{2}(p(n, d) + C_1(d, \alpha))$. By the Lemma and the induction hypothesis

$$C_1(d + 1, \alpha) = \prod 2^{C_i(d, \alpha)} = 2^{(1/2)(p(n, d) + q(n, d))} = q(n, d + 1).$$

For $\alpha \in K$,

$$C_1(0, \alpha) \leq n - 3 \quad \text{and} \quad C_1(d, \alpha) + 2 \sum_{i \geq 2} C_i(d, \alpha) \leq p(n, d).$$

An induction argument similar to the one above proves $C_1(d, \alpha) \leq r(n, d)$. This completes the proof.

COROLLARY 1. For $d \geq 2$,

$$N(n, d) = \frac{p(n, d)}{n!} + \frac{q(n, d)}{2(n - 2)!} + o\left(\frac{q(n, d)}{(n - 2)!}\right).$$

Proof. By the theorem, we need only prove $(n - 2)! r(n, d)/q(n, d) \rightarrow 0$ as $n \rightarrow \infty$. We use induction on d .

Routine calculation shows $r(n, 2) = (p(n, 2))^{(4 + \sqrt{2})/8}$ and $q(n, 2) = (p(n, 2))^{3/4}$. Therefore

$$\frac{(n - 2)! r(n, 2)}{q(n, 2)} = (n - 2)! (p(n, 2))^{(-2 + \sqrt{2})/8} \rightarrow 0$$

as $n \rightarrow \infty$.

Assuming the result for d , we prove it for $d + 1$.

$$\begin{aligned} \frac{(n - 2)! r(n, d + 1)}{q(n, d + 1)} &= (n - 2)! (2^{r(n, d) - q(n, d)})^{1/2} \\ &\leq (2^{(n-2)!r(n, d) - q(n, d)})^{1/2} \\ &\leq (2^{-q(n, d)})^{1/4}, \end{aligned}$$

for n sufficiently large, by the induction hypothesis. Clearly this converges to 0 as n gets large.

COROLLARY 2. For $d \geq 2$,

$$N(n, d) \sim p(n, d)/n!.$$

Proof. By Corollary 1, all that remains to be shown is that $n(n-1)q(n, d)/p(n, d) \rightarrow 0$ as $n \rightarrow \infty$. This may be done in a manner analogous to the proof of Corollary 1.

ACKNOWLEDGMENTS

I express my gratitude to Professor S. M. Ulam for originally suggesting this problem and to Professor Jan Mycielski for his helpful suggestions during the preparation of this note.

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