8D.1 POTENTIAL VORTICITY MIXING IN SHALLOW-WATER VORTICES

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1. INTRODUCTION

In a hurricane, the concentrated potential vorticity (PV) source due to heating in the eyewall can result in a reversal of the radial PV gradient, allowing the vortex to become barotropically unstable. In this manner an axisymmetric vortex can develop asymmetries, redistribute PV through chaotic nonlinear mixing, and eventually resymmetrize with a different, stable structure. Recent studies of this process have provided insight into diverse aspects of hurricane dynamics, including the development of spiral bands and mesoscale vortices, the existence of polygonal eyewalls, and asymmetric eye contraction (e.g., Guinn and Schubert, 1993; Montgomery and Kallenbach, 1997; Schubert et al., 1999).

While PV is conserved following the motion, nonlinear redistribution of PV results in filamentation which cannot be followed accurately by deterministic models with limited resolution. Consequently, various statistical approaches (such as minimum enstrophy and maximum entropy) have been proposed to compute equilibrium solutions. Most studies of this process to date have applied these methods in the simplest dynamical context, namely, two-dimensional nondivergent incompressible flow. This paper will concentrate on the extension of these ideas to the next level of dynamical complexity, namely, the shallow-water equations.

2. FORMULATION

The shallow-water equations can be written in rotational form as

$$\frac{\partial \mathbf{v}}{\partial t} + (\zeta + f)\mathbf{k} \times \mathbf{v} + \nabla B = 0,$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{v}h) = 0,$$

where $\mathbf{v}$ is the velocity, h is the free surface height, $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v}$ is the relative vorticity, $B = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + g(h - H)$ is the Bernoulli function, H is a constant (positive) reference height, f is the Coriolis parameter, and g is the gravitational constant. The corresponding PV, given by $q = H(\zeta + f)/h$, is conserved following the motion.

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We take the domain to be the unbounded (x, y) plane and assume that h is positive and bounded above and below. Furthermore, we assume that the flow of interest is localized, with $q = f$ outside a bounded region for all t, and tends toward geostrophic balance $f\mathbf{v} = g\mathbf{k} \times \nabla h$ as $r = \|\mathbf{r}\| \to \infty$. With these assumptions, it can be shown that the far-field solution satisfies

$$h = H + O\left(\sqrt{\frac{R}{r}} e^{-r/R}\right), \qquad r \to \infty,$$

where $R = \sqrt{gH}/f$ is the Rossby radius of deformation. This implies that the mass

$$M = \int (h - H),$$

energy

$$E = \int \frac{1}{2}\left[\mathbf{v} \cdot \mathbf{v}h + g(h - H)^2\right],$$

and angular momentum

$$L = \int \left[rvh + \frac{1}{2} f r^2 (h - H)\right]$$

are all conserved. Here we have written these quantities in terms of the deviation height $h - H$ so the integrals (over the whole domain) remain finite.

3. MAXIMUM ENTROPY

As the flow evolves in time, chaotic nonlinear mixing may cause any initial distribution of PV to become highly distorted on progressively finer scales, making deterministic solutions impractical. The Maximum Entropy approach (Miller, 1990; Robert and Sommeria, 1991) gives up on following these "microscopic states", and substitutes a statistical approach to find the most probable "macroscopic" equilibrium state (independent of t) to which the microscopic states converge. Here we formulate the Maximum Entropy problem, adapting the approach of Chavanis and Sommeria (2002) to the case of the unbounded f-plane and using a different formulation of the angular momentum constraint.

Suppose that initially the PV consists of patches of constant values, i.e., $q(\mathbf{r}, 0) \in \{\hat{q}_0, \ldots, \hat{q}_n\}$, with $\hat{q}_0 = f$. Since PV is conserved following the motion, in any microscopic state the PV in a small neighborhood of a point $\mathbf{r}$ will take on only these same values. Denoting the fraction of the area near $\mathbf{r}$ where $q = \hat{q}_l$ by $\rho_l(\mathbf{r})$, we can characterize the equilibrium state by the probabilistic distribution

$$\bar{q}(\mathbf{r}) = \sum_{l=0}^{n} \hat{q}_l \rho_l(\mathbf{r}), \qquad \sum_{l=0}^{n} \rho_l(\mathbf{r}) = 1$$

(the second equation is a normalization condition). The Maximum Entropy approach identifies the equilibrium state as that which maximizes the mixing entropy

$$S := -\sum_{l=0}^{n} \int \rho_l \ln(\rho_l) h$$

subject to the dynamical constraints of the system. By integrating hq over each patch we obtain circulation constraints, which we can write in terms of the mass as

$$M^{(l)} := \int \rho_l h = M_0^{(l)} := \int_{\Omega_l} h \quad (l = 1, \ldots, n),$$

where Ω_l is the initial region for patch l. Here we assume that each of these patches where $q \neq f$ is bounded; on the unbounded surrounding region Ω_0 where $q = \hat{q}_0 = f$ we obtain the constraint

$$M^{(0)} := \int (\rho_0 h - H) = M_0^{(0)} := \int_{\Omega_0} (h - H).$$

The Maximum Entropy problem consists of maximizing S subject to the constraints $M^{(l)} = M_0^{(l)}$ $(l = 0, \ldots, n)$, $E = E_0$, and $L = L_0$, and the far-field conditions $h \to H$ and $q \to f$ as $r \to \infty$.

The problem formulation is somewhat simpler in terms of the mass fraction $\sigma_l(\mathbf{r}) = h\rho_l(\mathbf{r})$. Then the normalization condition becomes

$$h = \sum_{l=0}^{n} \sigma_l,$$

the mass (circulation) constraints become

$$M^{(l)} = \int \sigma_l \quad (l = 1, \ldots, n), \qquad M^{(0)} = \int (\sigma_0 - H),$$

and the entropy functional reduces to

$$S = \int \left[h \ln(h) - \sum_{l=0}^{n} \sigma_l \ln(\sigma_l) \right].$$

In this form we can show that the entropy functional $S(\sigma)$ is strictly concave; likewise, the energy functional $E(\sigma)$ is strictly convex provided $u^2 + v^2 < gh$. The angular momentum functional $L(\sigma)$ is indefinite; however, the combination $E^*(\sigma) := E(\sigma) + \gamma L(\sigma)$ (where γ is a Lagrange multiplier) is strictly convex provided $u^2 + (v + \gamma r)^2 < gh$.

To solve for the equilibrium state, Chavanis and Sommeria (2002) introduce the mass-weighted streamfunction ψ and velocity potential ϕ satisfying

$$h\mathbf{v}^* = h(\mathbf{v} + \gamma \mathbf{k} \times \mathbf{r}) = H\mathbf{k} \times \nabla\psi + H\nabla\phi.$$

Then the equilibrium state is given by

$$\rho_l(r) = \frac{1}{Z} \exp(-\alpha_l + \beta \hat{q}_l \psi),$$

where α_l and β are Lagrange multipliers and

$$Z = \sum_{l=0}^{n} \exp(-\alpha_l + \beta \hat{q}_l \psi),$$

with the corresponding PV given by

$$\bar{q} = \frac{\sum_{l=0}^{n} \hat{q}_l \exp(-\alpha_l + \beta \hat{q}_l \psi)}{\sum_{l=0}^{n} \exp(-\alpha_l + \beta \hat{q}_l \psi)}.$$

Note that Z and $\bar{q}$ are functions of ψ only, with

$$\bar{q} = \frac{1}{\beta} \frac{d \ln(Z)}{d\psi}.$$

The equilibrium solution has $\phi = 0$, which results in the balance equation

$$h\mathbf{v}^* = H\mathbf{k} \times \nabla\psi.$$

In the case of axisymmetric flow, this reduces to gradient balance:

$$\left(f + \frac{v}{r} \right) v = g \frac{dh}{dr}.$$

To compute this equilibrium solution we extend the iterative algorithm of Turkington and Whitaker (1996), using the convexity properties established above to guarantee convergence. Results of these computations for hurricane-like vortices will be presented at the conference.

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