P1.49

COMPARISON OF DISCRETIZATION ACCURACY IN AN ADAPTIVE MULTIGRID BAROTROPIC TROPICAL CYCLONE TRACK MODEL

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1. INTRODUCTION

Among the many sources of error in modeling tropical cyclones is discretization error: how well are the governing equations approximated numerically? While discretization error is typically small, numerical methods with smaller errors should be preferred over others, all else being equal. This paper addresses the tradeoff between higher-accuracy discretization and increased computational work in an adaptive multigrid tropical cyclone track model.

2. MODEL DESCRIPTION

The model described here is a higher-order version of the MUDBAR model described in Fulton (1997, 2000). Formulated on a section of the sphere using a Mercator projection (true at latitude $\phi = \phi_c$), the model consists of the modified barotropic vorticity equation

$$\frac{\partial \zeta}{\partial t} + m^2 J(\psi, \zeta) + \beta m \frac{\partial \psi}{\partial x} = \nu m^2 \nabla^2 \zeta, \qquad (1)$$

with relative vorticity ζ and streamfunction ψ related by

$$L\psi := \left(\nabla^2 - \frac{\gamma^2}{m^2} \right) \psi = \frac{\zeta}{m^2}. \qquad (2)$$

Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $J(\psi, \zeta)$ is the Jacobian of (ψ, ζ) with respect to (x, y), $\beta = 2\Omega \cos\phi / a$ (with a and Ω the radius and rotation rate of the earth), $m = \cos\phi_c / \cos\phi$ is the map factor, ν is the diffusion coefficient, and γ is the inverse of the effective Rossby radius. The model domain is a rectangle in x and y centered at $(x, y) = (0, 0)$, where $(\lambda, \phi) = (\lambda_c, \phi_c)$. At the boundaries we specify the streamfunction ψ (and thus the normal component of the velocity); where there is inflow, we also specify the vorticity ζ.

The equations are discretized in space by finite differences on uniform rectangular grids, approximating the advection terms by the Arakawa Jacobian. The model achieves higher resolution near the vortex by superimposing nested overlapping grids with different mesh sizes. Unlike conventional nested-grid methods, using multigrid processing (Brandt, 1977) in solving for the streamfunction allows optimal solution speed and accurate estimates of truncation error; the latter can be used in an adaptive mesh refinement scheme to provide just the resolution needed at each point. More detail for the second-order model is given in Fulton (2000); here we investigate the gains possible by using fourth-order differencing.

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3. FOURTH-ORDER DISCRETIZATION

The model is discretized in time using the classical fourth-order Runge-Kutta scheme (also used in the second-order model). To achieve fourth-order discretization in space we use the methods described below.

3.1 *Streamfunction equation solution*

For the Helmholtz problem (2) we use a compact fourth-order discretization known as the Mehrstellen Verfahren (MV) discretization (e.g., Schaffer, 1984). For the Poisson problem (i.e., setting $\gamma = 0$ and $m = 1$) this takes the form

$$\frac{1}{6h^2} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \psi^h = \frac{1}{12} \begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} \zeta^h. \qquad (3)$$

Here ψ^h and ζ^h denote grid functions consisting of the approximate values of streamfunction and vorticity on a uniform rectangular grid with mesh spacing h in x and y, and the square brackets denote difference stencils on that grid. Since this compact discretization involves only the nearest-neighbor points, it can be applied at all interior points on the grid (with Dirichlet boundary conditions specifying the boundary values).

To solve the resulting discrete problem efficiently we use a multigrid method. As in the second-order model, this uses the Full Multigrid (FMG) approach, solving on each grid level from coarsest to finest in turn, interpolating the solution from the previous level to provide the initial approximation on the new level. The fourth-order discretization (3) is used only on the "currently finest" level. Compared to the second-order method, three changes are needed:

(1) Gauss-Seidel relaxation is used with lexicographic ordering (rather than red-black);
(2) Two V(2,1) cycles are needed to solve on each level (rather than one V(1,1) cycle);
(3) The initial interpolation to a new level requires sixth-order accuracy (rather than fourth-order bicubic interpolation).

The change in relaxation ordering is a result of using a nine-point stencil; the other two changes are required to ensure that the discrete problem is solved adequately on any given grid and that its solution is accurately approximated on the next grid. With these changes we find that the resulting method [in multigrid terminology, a 2-FMV(2,1) method] solves to the level of truncation error (i.e., residual norm less than the truncation error norm) for problems with known (smooth) solutions, independent of the mesh size h. Numerical tests verify that the error in the computed solution of the Helmholtz problem is indeed $O(h^4)$.

3.2 *Vorticity equation discretization*

The standard second-order Jacobian of Arakawa (1966) uses only nearest-neighbor points and conserves discrete analogues of vorticity, enstrophy, and kinetic energy. Unfortunately, it can be shown that no fourth-order Jacobian has the same properties: to achieve fourth-order accuracy while maintaining the conservation properties one must include points other than the nearest neighbors. Thus, we use the fourth-order Jacobian from the Appendix of Arakawa (1966) at all interior points on the grid except those along the first interior lines (adjacent to each boundary), where we use the usual second-order Jacobian. Since the set of points where lower accuracy is used is small, this has negligible effect on the overall accuracy. Numerical tests with specified ψ and ζ verify that the truncation error for the Jacobian is indeed $O(h^4)$.

4. NUMERICAL RESULTS

To quantify the model performance we use the track error, defined as the difference in vortex position compared to a high-resolution reference run ($h = 4\,\text{km}$). Figure 1 shows the track error as a function of h for several uniform-grid runs of the second-order and fourth-order models at 24, 48, and 72 hours and the mean errors over each 72 hour model run. The slopes of the curves verify that the overall model converges at approximately the proper rate.

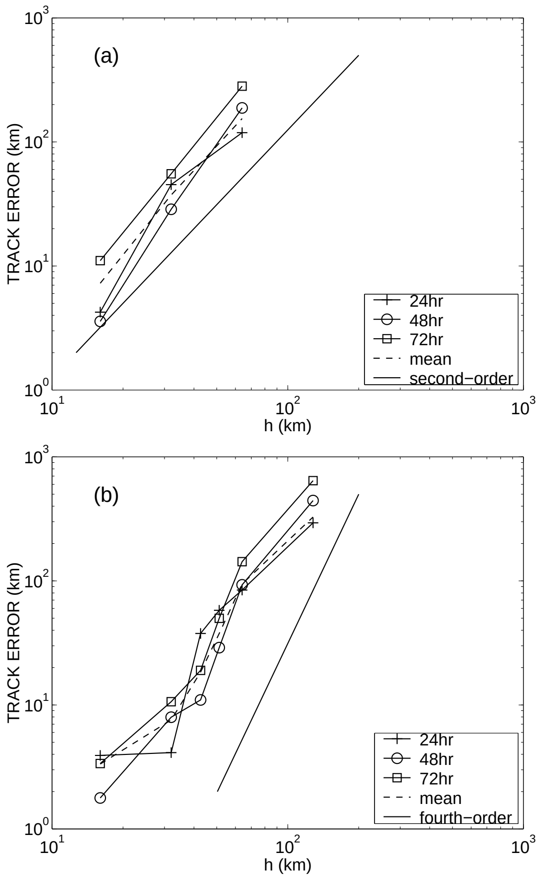

Figure 1. Overall convergence of the model: (a) second-order version, (b) fourth-order version.

Does the asymptotic superiority of fourth-order differencing pay in practice? To answer this, we ran both models with variety of grids (uniform grids and local refinements using patches of various specified sizes). Figure 2 shows the resulting mean track errors as a function of the computer time (for a 72 hour model run). For the uniform-grid runs (points joined by lines in Fig. 2) the fourth-order model is approximately 5–10 times faster than the second-order model (for the same level of error); for the same amount of computer time, it is at least 5 times more accurate. With local refinement (remaining points in Fig. 2), fourth-order differencing shows some advantage for moderate resolution (e.g., errors of 10–30 km); however, the advantage apparently disappears at higher resolutions. This latter result may reflect the need for higher-order grid transfers in the fourth-order model with local refinement.

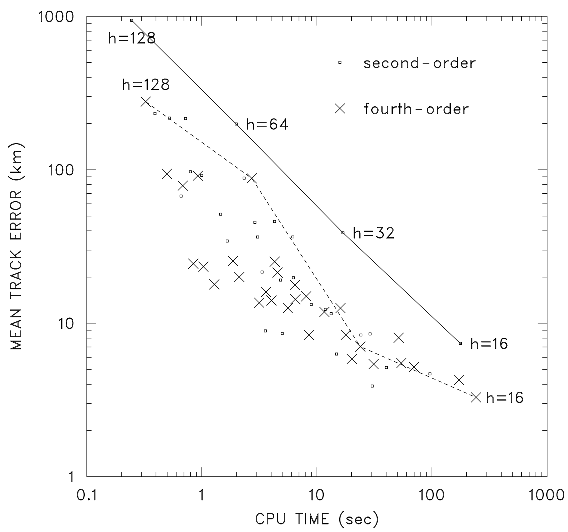

Figure 2. Track error vs. CPU time.

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