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An analytical method for linear elliptic PDEs and its numerical implementation

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Abstract

A new numerical method for solving linear elliptic boundary value problems with constant coefficients in a polygonal domain is introduced. This method produces a generalized Dirichlet–Neumann map: given the derivative of the solution along a direction of an arbitrary angle to the boundary, the derivative of the solution perpendicular to this direction is computed without solving on the interior of the domain. If desired, the solution on the interior can then be computed via an integral representation.

The key to the method is a "global condition" which couples known and unknown components of the derivative on the boundary and which is valid for all values of a complex parameter k. This condition has been solved recently analytically for several equations on simple domains. In this paper, first the previous analytical result is strengthened, and then a numerical method is introduced for solving the global condition for the Laplace equation on an arbitrary bounded convex polygon. Numerical results demonstrate the applicability and convergence of the method; however, a rigorous proof of convergence remains open. Extensions to other problems are also discussed.

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Keywords: Elliptic partial differential equations; Dirichlet–Neumann map; Global condition; Spectral method

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1. Introduction

Solving boundary value problems for partial differential equations (PDEs) is a central topic in applied mathematics. Analytical methods (e.g., separation of variables and transform techniques) are valued for their exactness and the insight they provide; however, the range of problems they solve is limited. Numerical methods (e.g., finite element, finite difference, and spectral methods) solve a much wider range of problems, albeit only approximately. Some methods (e.g., boundary integral methods) combine specific analytical information about the solution with numerical approximations. This paper describes a new method which fits into this latter class.

Recently, a new analytical method for studying boundary value problems for integrable PDEs in two dimensions has been introduced in the literature, see for example [1,2]. This method has been applied to linear elliptic PDEs in convex polygonal domains [2], yielding analytical solutions in cases not amenable to treatment by standard transform methods. A key to this method is the *global condition* [cf. (7)] which couples specified and unknown values of the solution or its derivatives on the boundary. In cases where this condition can be solved analytically, the method yields the solution in closed form, generalizing standard transform methods (e.g., Fourier, Bessel, etc.).

In this paper, we extend this method in two ways. First, we strengthen the analytical results of [2]: we show that the global condition is not only a necessary but also a sufficient condition for existence. This reduces the problem of solving Laplace's equation to the problem of solving the global condition. Second, we introduce an approach to solving the global condition numerically. The result is a new method for solving PDEs in two dimensions which couples the analytical information obtained by the method of [2] with the numerical solution of a one-dimensional problem. Applied to the Laplace problem on a convex polygonal domain, the method provides a generalized Dirichlet–Neumann map: given the derivative of the solution in the direction of some (arbitrary) angle to the boundary, the method yields the derivative of the solution perpendicular to this direction *without* solving on the interior of the domain.

This paper is organized as follows: Section 2 outlines the analytical method of [2] and gives an overview of the numerical implementation introduced here. In that section, we also discuss the relation of our method to a boundary integral method (the boundary element method). In Section 3 we extend the analytical result of [2], proving (as mentioned above) that solving the Laplace equation is equivalent to solving the global condition. The main improvement as compared with [2] is that now the result is proven *without* assuming the existence of the solution. Section 4 gives the details of the numerical solution of the global condition. In Section 5 we apply the method to the Laplace equation in a variety of domains with different boundary conditions and present numerical results demonstrating its convergence. Section 6 summarizes our conclusions and discusses extensions.

2. Overview

In this section, we review the analytical method of [2], outline the numerical solution of the resulting global condition, and compare our approach with boundary integral methods, with which it shares some similarities.

2.1. The method of Fokas

The method introduced by Fokas [1,2] solves integrable PDEs in two dimensions. An equation in two dimensions (x, y) is called integrable if and only if it can be expressed as the condition that a certain associated differential 1-form $W(x, y, k)$, $k \in \mathbb{C}$, is closed, i.e., $dW = 0$. Examples of integrable equations are linear PDEs with constant coefficients and the usual integrable nonlinear PDEs such as the nonlinear Schröndinger and the Korteweg–de Vries equations. In what follows, we formulate the global condition without using the language of differential forms.

For elliptic equations it is convenient to replace the usual Cartesian coordinates (x, y) with the complex coordinates $(z, \bar{z}) = (x + iy, x - iy)$. For example, the Helmholtz equation

$$q_{xx} + q_{yy} + 4\alpha q = 0, \quad \alpha \text{ constant}, \tag{1}$$

may be written, using $q_z = \frac{1}{2}(q_x - iq_y)$ and $q_{\bar{z}} = \frac{1}{2}(q_x + iq_y)$, as

$$q_{z\bar{z}} + \alpha q = 0. \tag{2}$$

This equation can be rewritten in the form

$$(e^{-ikz - i\alpha\bar{z}/k} q_z)_{\bar{z}} + \frac{i\alpha}{k}(e^{-ikz - i\alpha\bar{z}/k} q)_z = 0, \quad k \in \mathbb{C}. \tag{3}$$

It is emphasized that (2) is equivalent to (3) for an arbitrary complex parameter k.

Suppose that $q(z, \bar{z})$ satisfies (2) in a simply connected bounded domain D with boundary ∂D. Then the complex form of Green's theorem implies

$$\int_{\partial D} W(z, \bar{z}, k) = 0, \quad k \in \mathbb{C}, \tag{4}$$

where

$$W(z, \bar{z}, k) = e^{-ikz - i\alpha\bar{z}/k}\left[q_z\, dz - \frac{i\alpha}{k} q\, d\bar{z}\right]. \tag{5}$$

Likewise, the Laplace equation, i.e., (2) with $\alpha = 0$, is equivalent to

$$(e^{-ikz} q_z)_{\bar{z}} = 0, \quad k \in \mathbb{C} \tag{6}$$

so in analogy with (4) we now find

$$\int_{\partial D} e^{-ikz} q_z\, dz = 0, \quad k \in \mathbb{C}. \tag{7}$$

Following [2] we shall refer to (4) and (7) as the *global conditions* associated with the Helmholtz and Laplace equations, respectively.

For example, suppose that $q(z, \bar{z})$ satisfies the Laplace equation in a convex bounded polygon with vertices $z_1, z_2, \ldots, z_n$ (indexed counterclockwise, modulo n) and interior D as in Fig. 1. Then the global condition (7) becomes

$$\sum_{j=1}^{n} \rho_j(k) = 0, \quad k \in \mathbb{C}, \tag{8}$$

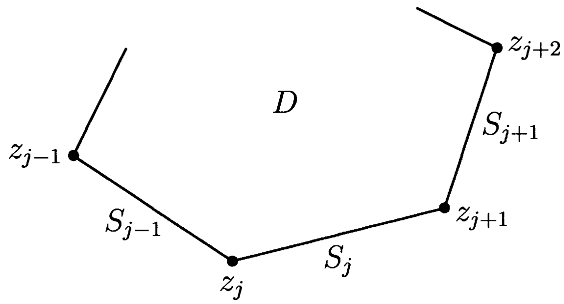

Fig. 1. Part of the bounded convex polygon with vertices z_j, sides S_j, and interior D.

where the functions $\rho_j(k)$ are defined by the line integrals

$$\rho_j(k) = \int_{S_j} e^{-ikz} q_z(z)\,dz, \quad k \in \mathbb{C}, \; j = 1, \ldots, n \tag{9}$$

with S_j being the side from z_j to z_{j+1} (not including the endpoints).

It was shown in [2] that the global condition plays a crucial role in the analysis of boundary value problems. For example, consider the Neumann problem for the Laplace equation in the above polygon. Let $q_s^{(j)}$ and $q_n^{(j)}$ denote the tangential and (outward) normal components of q_z along the side S_j. Then on this side

$$q_z = \tfrac{1}{2} e^{-i\alpha_j}(q_s^{(j)} + i q_n^{(j)}), \quad \alpha_j = \arg(z_{j+1} - z_j). \tag{10}$$

Substituting (10) into (9), the global condition becomes

$$\sum_{j=1}^{n} e^{-i\alpha_j} \psi_j(k) = G(k), \quad k \in \mathbb{C}, \tag{11}$$

where $\psi_j(k)$ denotes the unknown line integral

$$\psi_j(k) = \int_{S_j} e^{-ikz} q_s^{(j)}(z)\,dz, \quad k \in \mathbb{C}, \; j = 1, \ldots, n \tag{12}$$

and $G(k)$ can be computed in terms of the given boundary data $q_n^{(j)}$. Eq. (11) is only one equation for the n unknown functions $\psi_j(k)$. In spite of this ominous-looking situation, it is possible using the global condition (11) to determine *all* the unknown functions $\psi_j(k)$. This is a consequence of the fact that (11) is valid for *all complex values* k. The analytical investigation of the global condition is discussed in [2] in general, and in [3] for the Laplace equation in particular. One of the main goals of this paper is to introduce a numerical algorithm for solving the global condition.

Eq. (11) indicates that the global condition determines the integrals $\psi_j(k)$ and *not* the functions $q_s^{(j)}$. This suggests that it would be desirable to express $q(z,\bar{z})$ in terms of $\rho_j(k)$ and not in terms of the boundary values of q. For linear PDEs such formulae have recently been derived using the spectral method introduced in [2]. For example, for the Laplace equation the following result from [2] is valid:

Proposition 1. *Consider the Laplace equation in a convex bounded polygon with vertices* $z_1, z_2, \ldots, z_n$ (*indexed counterclockwise, modulo* n) *and interior* D. *Assume that appropriate boundary conditions are prescribed such that there exists a solution* $q(z,\bar{z})$ *which is smooth all the way to the boundary. Then*

$$q_z = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{\ell_j} e^{ikz} \rho_j(k)\, dk, \quad z \in D, \tag{13}$$

where $\rho_j(k)$ *are the functions defined by* (9) *and* ℓ_j *are the rays in the complex* k*-plane oriented away from the origin defined by*

$$\ell_j = \{k \in \mathbb{C}: \arg(k) = -\arg(z_{j+1} - z_j)\}. \tag{14}$$

In Section 2 we will prove a stronger version of this proposition (cf. Proposition 2). This result reduces the solution of a given boundary value problem for the Laplace equation to the following problem: Use the global condition (8) to determine $\rho_j(k)$ in terms of the given boundary data.

In this paper, we will study the Laplace equation for an arbitrary convex bounded polygon with an arbitrary component of the derivative specified on each side. Specifically, on each side S_j we will specify the derivative in the direction given by the angle δ_j relative to the positive real axis (angle $\beta_j = \alpha_j - \delta_j$ measured outward from the side S_j) leading to the mixed boundary condition

$$\cos(\beta_j) q_s^{(j)} + \sin(\beta_j) q_n^{(j)} = g^{(j)}, \quad z \in S_j, \tag{15}$$

where $g^{(j)}$ is a given smooth function. Dirichlet and Neumann conditions correspond to the special cases $\beta = 0$ and $\pi/2$, respectively. For this problem the relevant unknown is the derivative in the direction normal to δ_j, i.e., the function $f^{(j)}$ defined by

$$-\sin(\beta_j) q_s^{(j)} + \cos(\beta_j) q_n^{(j)} = f^{(j)}, \quad z \in S_j. \tag{16}$$

Solving (15) and (16) for $q_s^{(j)}$ and $q_n^{(j)}$ and substituting into (10) yields

$$q_z^{(j)} = \tfrac{1}{2} e^{-i\delta_j} (g^{(j)} + i f^{(j)}). \tag{17}$$

Replacing q_z in (9) by the above expression, it follows that $\rho_j(k)$ involves the unknown integral

$$\psi_j(k) = \int_{S_j} e^{-ikz} f^{(j)}(z)\, dz. \tag{18}$$

2.2. Numerical solution

In order to determine these unknown integrals, we use a collocation projection of the global condition in the complex k-plane (see Section 4.4):

(1) For each side S_j, set $F^{(j)} = f^{(j)} - f_*^{(j)}$, where the linear function $f_*^{(j)}$ is chosen so that $F^{(j)}$ vanishes at the endpoints z_j and z_{j+1}. This is made possible by using the continuity of q_z at the n vertices $z_1, \ldots, z_n$ to determine the values of $f^{(j)}$ at the endpoints.

(2) Approximate $F^{(j)}$ by

$$F_N^{(j)}(s) = \sum_{r=1}^{N} c_r^{(j)} \varphi_r(s), \tag{19}$$

where s is a local parameter along the side S_j, $\{\varphi_r\}_{r=1}^N$ are appropriate basis functions, and N is even. This approximation introduces $n \times N$ unknown (real) constants $c_r^{(j)}$, $1 \leqslant r \leqslant N$, $1 \leqslant j \leqslant n$.

(3) Evaluate the global condition (8) at $M = N/2$ collocation points on each ray $\hat{\ell}_j$, where $\hat{\ell}_j$ is the continuation of ℓ_j, i.e.,

$$\hat{\ell}_j = \{k \in \mathbb{C} : \arg(k) = \pi - \alpha_j\}, \quad j = 1, \ldots, n. \tag{20}$$

The reason for this choice of the rays $\hat{\ell}_j$ is explained in Section 4.2. This yields $n \times M$ (complex) linear equations.

(4) Solve the resulting linear system to obtain the constants $c_r^{(j)}$, which in turn yield an approximation to $f^{(j)}(s)$ and thus $\rho_j(k)$, $j = 1, \ldots, n$.

Note that while this method uses a collocation projection and a spectral representation, it is not a typical "spectral collocation" method: the collocation takes place in the complex k-plane, and the spectral representation used refers not to the basis functions in the numerical approximation but to the underlying analytical representation.

2.3. Discussion

Since the method treated here shares some similarities with boundary integral methods, it is appropriate to discuss their relation. We will restrict our comments to the boundary element method (BEM) [8], which is a well-established method for the numerical solution of boundary value problems; its applicability and underlying theory have been (and still are) studied extensively (see, e.g., [10] and the references therein). When using the BEM to approximate the solution to elliptic boundary value problems, one starts with a *fundamental solution* and converts the given PDE into an integral equation posed on the boundary of the domain. The resulting equation is then discretized and solved numerically. This procedure reduces the dimension of the problem by one, hence keeping the computational cost low. The use of a fundamental solution can be viewed as a disadvantage, since its availability and/or simplicity is not always guaranteed. As a result, some researchers have combined the BEM with other methods to bypass this step, while still solving a problem on the boundary of the domain (e.g., [6,11]).

Like the BEM, the method presented in this paper combines analytical information with a numerical approximation and reduces the numerical work to solving a one-dimensional problem posed on the boundary of the domain. However, this is the *only* feature these methods share. In the present method, the one-dimensional problem to be solved (the global condition) comes from a different source and is solved differently. Furthermore, the resulting functions $\rho_j(k)$ then provide the solution of the original PDE in a different form (the spectral representation (13)). Also, while strictly Dirichlet–Neumann maps have been studied in the context of the BEM (e.g., [4,9]), we believe that the ability of the present method to produce automatically a *generalized* Dirichlet–Neumann map is not present in the BEM, nor in boundary integral methods in general.

3. Spectral solution of the Laplace equation

Here, we state and prove a stronger version of Proposition 1 for the solution of the Laplace equation. As above, we let $z_1,\dots,z_n$ denote the vertices of a convex bounded polygon in the complex plane (indexed counterclockwise, modulo n) with interior D; S_j denotes the side from z_{j+1} to z_j (not including the endpoints), and $\alpha_j := \arg(z_{j+1}-z_j)$ denotes the angle between side S_j and the positive real axis. Note that the boundary ∂D of D consists of $S := \bigcup_{j=1}^{n} S_j$, together with the vertices $z_1,\dots,z_n$. The following proposition allows for the case of singularities at the vertices, and establishes (rather than assumes) the existence of the solution.

Proposition 2. *For each $j=1,\dots,n$ let $r^{(j)} \in H^{1/2+\varepsilon}(S_j)$ for $\varepsilon>0$ with $r^{(j)}(z_{j+1}) = r^{(j+1)}(z_{j+1})$ and define $\rho_j(k)$ by the line integral*

$$\rho_j(k) = \int_{S_j} e^{-ikz} r^{(j)}(z)\,dz, \quad k \in \mathbb{C} \tag{21}$$

along that side. Assume that the functions ρ_j satisfy the global condition (8). Then the function

$$r(z) := \frac{1}{2\pi}\sum_{j=1}^{n}\int_{\ell_j} e^{ikz}\rho_j(k)\,dk \tag{22}$$

and its antiderivative $q(z)$ are continuous on $D\cup S$ and analytic on D, $Re(q)$ satisfies the Laplace equation on D, and on each side S_j $(j=1,\dots,n)$, $q_z = r = r^{(j)}$.

Thus, solving the Laplace equation is equivalent to solving the global condition: given appropriate boundary data, amounting to “half” of q_z, if one can find the other half by the requirement that the functions ρ_j defined by (21) satisfy the global condition, then the function $r(z)$ defined by (22) solves the Laplace equation and satisfies the given boundary conditions.

The key to the proof is the fact that for certain values of z and k, the integrand in (22) decays exponentially as $|k|\to\infty$. More precisely, we have:

Lemma 1. *For $k\in\ell_j$ and $z\in(D\cup S)-S_j$, the function*

$$e^{ikz}\rho_j(k) = \int_{S_j} e^{ik(z-z')} r^{(j)}(z')\,dz' \tag{23}$$

decays exponentially as $|k|\to\infty$.

Proof. By convexity (see Fig. 2),

$$\alpha_j < \arg(z-z_j) < \arg(z-z') < \arg(z-z_{j+1}) < \alpha_j+\pi, \tag{24}$$

so there exists some $\varepsilon\in(0,\pi/2)$ such that

$$\alpha_j+\varepsilon \leqslant \arg(z-z') \leqslant \alpha_j+\pi-\varepsilon \quad \forall z'\in S_j. \tag{25}$$

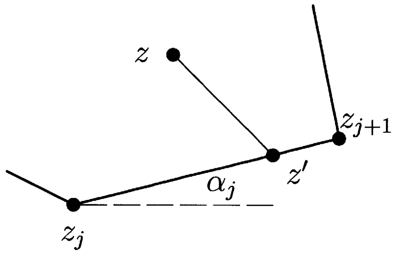

Fig. 2. Geometry for Lemma 1.

Since $\arg(k) = -\alpha_j$, we have

$$\varepsilon \leqslant \arg[k(z - z')] \leqslant \pi - \varepsilon \quad \forall z' \in S_j. \tag{26}$$

Thus, $\operatorname{Im}[k(z - z')] \geqslant |k|\eta \sin(\varepsilon)$, where $\eta := \inf_{z' \in S_j} |z - z'| > 0$, hence

$$|e^{ik(z-z')}| \leqslant e^{-|k|\eta \sin(\varepsilon)} \quad \forall z' \in S_j. \tag{27}$$

This gives the bound

$$|e^{ikz}\rho_j(k)| = \left| \int_{S_j} e^{ik(z-z')} r^{(j)}(z')\, dz' \right| \leqslant e^{-|k|\eta \sin(\varepsilon)} \|r^{(j)}\|_{L^1(S_j)}. \qquad \square \tag{28}$$

Likewise, for z on a side other than S_j, the exponential decay of $e^{ikz}\rho_j(k)$ between two associated rays allows us to change paths of integration as follows:

Lemma 2. *If $z \in S_p$ with $p \neq j$, then*

$$\int_{\hat{\ell}_p} e^{ikz}\rho_j(k)\, dk = \int_{\ell_j} e^{ikz}\rho_j(k)\, dk. \tag{29}$$

Proof. When $|\alpha_j - \alpha_p| = \pi$ (i.e., sides S_j and S_p are parallel) there is nothing to prove, since the rays ℓ_j and $\hat{\ell}_p$ coincide. Thus, we consider the case $0 < \alpha_p - \alpha_j < \pi$ as shown in Fig. 3(a); the case $0 < \alpha_j - \alpha_p < \pi$ can be treated similarly. Note that if $p \neq j + 1$ then the integrand decays exponentially for all k on and between the rays ℓ_j and $\hat{\ell}_p$ and the proof is straightforward. Therefore, Fig. 3(a) depicts the case $p = j + 1$, which requires the more involved argument outlined below.

Consider the contour $C_1 \cup C_2 \cup C_3 \cup C_4$ in the complex k-plane as shown in Fig. 3(b), where $\varepsilon \in (0, \pi - \alpha_p + \alpha_j)$. Since from (21) and the Schwarz inequality $\rho_j(k)$ is an entire function of k, by the Cauchy–Goursat theorem

$$-\int_{C_1} e^{ikz}\rho_j(k)\, dk = \int_{C_2} e^{ikz}\rho_j(k)\, dk + \int_{C_3} e^{ikz}\rho_j(k)\, dk + \int_{C_4} e^{ikz}\rho_j(k)\, dk. \tag{30}$$

In the limit as $R \to \infty$, (30) reduces to (29) provided that the last two integrals vanish. To bound the integral along C_3, note that the integrand satisfies a uniform bound of the form (28) as in Lemma 1, with $|k| = R$ for k on C_3. Thus, the integral along C_3 vanishes in the limit as $R \to \infty$.

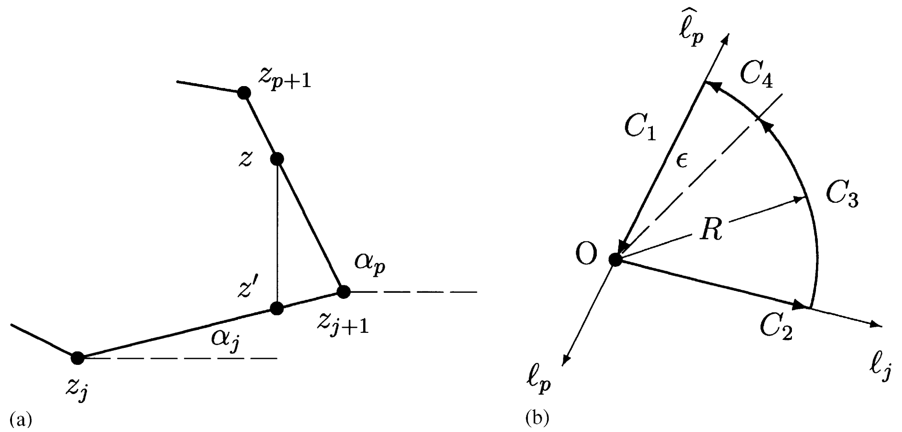

Fig. 3. Geometry for Lemma 2 in (a) the z-plane and (b) the k-plane.

To bound the integral along C_4, we write

$$\int_{C_4} e^{ikz}\rho_j(k)\,dk = \int_{C_4} e^{ik(z-z_{j+1})}g(k)\,dk, \tag{31}$$

where

$$g(k) := e^{ikz_{j+1}}\rho_j(k) = \int_{S_j} e^{ik(z_{j+1}-z')}r^{(j)}(z')\,dz'. \tag{32}$$

For ε sufficiently small we can establish the uniform bound

$$|g(k)| \leqslant B_R := \|e^{-R|z_{j+1}-z'|\sin(\varepsilon)}\|_{L^2(S_j)}\|r^{(j)}\|_{L^2(S_j)} \tag{33}$$

and then (as in the proof of the Jordan Lemma) we have

$$\left|\int_{C_4} e^{ikz}\rho_j(k)\,dk\right| \leqslant \frac{\pi B_R}{r}(1-e^{-rR}), \tag{34}$$

so the integral along C_4 vanishes in the limit as $R\to\infty$ since $B_R\to 0$. □

Proof of Proposition 2. First, for $k\in\ell_j$, $\rho_j(k)$ is a scaled version of the Fourier transform of $r^{(j)}$. To see this, parameterize $z\in S_j$ as $z=m_j+sh_j$, $-\pi<s<\pi$, where $m_j:=(z_j+z_{j+1})/2$ and $h_j:=(z_{j+1}-z_j)/(2\pi)$, and set

$$f(s)=\begin{cases} r^{(j)}(m_j+sh_j), & -\pi<s<\pi,\\ 0 & \text{otherwise.}\end{cases} \tag{35}$$

Likewise, parameterize $k\in\ell_j\cup\hat{\ell}_j\cup\{0\}$ as $k=t/h_j$, $t\in\mathbb{R}$, so positive and negative t correspond to $k\in\ell_j$ and $k\in\hat{\ell}_j$, respectively. For these k values, (21) reduces to

$$\rho_j(k)=\rho_j(t/h_j)=2\pi h_j e^{-itm_j/h_j}\hat{f}(t)\quad \forall t\in\mathbb{R}, \tag{36}$$

where

$$\hat{f}(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-its}f(s)\,ds \tag{37}$$

is the Fourier transform of f. The inverse transform is

$$f(s)=\int_{-\infty}^{\infty} e^{its}\hat{f}(t)\,dt, \tag{38}$$

which converges for all $s\in(-\pi,\pi)$. This may be written as

$$r^{(j)}(z)=\frac{1}{2\pi}\left[\int_{\ell_j} e^{ikz}\rho_j(k)\,dk-\int_{\hat{\ell}_j} e^{ikz}\rho_j(k)\,dk\right]\quad \forall z\in S_j \tag{39}$$

with both integrals finite.

Now since for each $j=1,\dots,n$ the integral in (22) is finite for $z\in(D\cup S)-S_j$ by Lemma 1 and for $z\in S_j$ by (39), the function $r(z)$ is defined for all $z\in D\cup S$. By subtracting a polynomial we may assume that $r^{(j)}=0$ at the endpoints of S_j and hence $f\in H^{1/2+\varepsilon}(\mathbb{R})$. Thus, $\hat{f}\in L^1(\mathbb{R})$ so r is continuous. To show that r is analytic on D, we can differentiate formally with respect to z; the resulting integrals can be shown using Lemma 1 to converge uniformly on a neighborhood of any point $z\in D$, thus justifying the differentiation. Any antiderivative q of r is also continuous on $D\cup S$ and analytic on D, and since $q_{z\bar{z}}=r_{\bar{z}}=0$, Re$(q)$ satisfies the Laplace equation on D.

Finally, to show that r matches the prescribed boundary values, we fix $p\in\{1,\dots,n\}$ and $z\in S_p$, multiply the global condition (8) by e^{ikz} and integrate over $\hat{\ell}_p$ to obtain

$$\int_{\hat{\ell}_p}\sum_{j=1}^{n} e^{ikz}\rho_j(k)\,dk=0. \tag{40}$$

Since each of the integrals is finite, we can interchange the order of integration and summation to obtain

$$\sum_{j=1}^{n}\int_{\hat{\ell}_p} e^{ikz}\rho_j(k)\,dk=0. \tag{41}$$

Dividing by 2π and subtracting from (22) yields

$$r(z)=\frac{1}{2\pi}\sum_{j=1}^{n}\left[\int_{\ell_j} e^{ikz}\rho_j(k)\,dk-\int_{\hat{\ell}_p} e^{ikz}\rho_j(k)\,dk\right]. \tag{42}$$

For $j\neq p$ the two integrals cancel by Lemma 2, leaving

$$r(z)=\frac{1}{2\pi}\left[\int_{\ell_p} e^{ikz}\rho_p(k)\,dk-\int_{\hat{\ell}_p} e^{ikz}\rho_p(k)\,dk\right]=r^{(p)}(z),\quad z\in S_p, \tag{43}$$

where the last step follows from (39). □

4. Numerical solution of the global condition

It follows from Proposition 2 that the key to solving the Laplace problem is solving the global condition (8) for $\rho_j(k)$ $(j=1,\dots,n)$ in terms of the given boundary data. When this can be solved analytically [2], the resulting solution is given by (13). In this section we give the details of the numerical method outlined in Section 2.2 for solving the global condition.

4.1. Parameterization

To write the global condition in a form appropriate for numerical solution, we first parameterize z on the side S_j by $z = m_j + sh_j$, $-\pi < s < \pi$, where $m_j := (z_j + z_{j+1})/2$ and $h_j := (z_{j+1} - z_j)/(2\pi)$, as in the proof of Proposition 2. Then using (17)—and reinterpreting $f^{(j)}$ and $g^{(j)}$ as functions of s rather than z—we can write (9) in the form

$$\rho_j(k) = \pi h_j e^{-i\delta_j} e^{-ikm_j} \left[\hat{g}^{(j)}(kh_j) + i\hat{f}^{(j)}(kh_j)\right], \tag{44}$$

where the hat denotes the Fourier transform [cf. (37)], now evaluated at complex arguments. Thus, the global condition (8) takes the form

$$\sum_{j=1}^{n} h_j e^{-i\delta_j} e^{-ikm_j} \hat{f}^{(j)}(kh_j) = i \sum_{j=1}^{n} h_j e^{-i\delta_j} e^{-ikm_j} \hat{g}^{(j)}(kh_j), \tag{45}$$

where the functions $\hat{f}^{(j)}$ are the unknowns, corresponding to the unknown integrals ψ_j in Section 2.1 [cf. (18)].

4.2. The choice of k

The global condition holds for all $k \in \mathbb{C}$. Which values should we use? Eq. (13) indicates that $\rho_j(k)$ is needed for k on the ray ℓ_j defined by (14), where by construction $\rho_j(k)$ is oscillatory in k. Indeed, the term $e^{-ikh_j s}$ in the integrand of $\rho_j(k)$ is oscillatory (and thus bounded) for all $s \in (-\pi, \pi)$ if and only if kh_j is real. Now while kh_j is real for k on the ray ℓ_j, on this ray the term e^{-ikm_j} multiplying the unknown $\hat{f}^{(j)}(k)$ in the global condition (45) is exponentially small as $|k| \to \infty$, so these unknowns will be only weakly coupled. In contrast, kh_j is also real for k on the ray $\hat{\ell}_j$ defined by (20). On this ray the term e^{-ikm_j} is exponentially large as $|k| \to \infty$, so the unknowns $\hat{f}^{(j)}(k)$ will be strongly coupled by the global condition (45).[3] Consequently, to derive a well-conditioned system of equations we choose k on the ray $\hat{\ell}_j$.

To obtain such a system, we choose a side index $p \in \{1, 2, \ldots, n\}$ and set $k = -l/h_p$ in the global condition (45); then positive and negative l correspond to $k \in \hat{\ell}_p$ and $k \in \ell_p$, respectively. Scaling the result by the coefficient of the term $j = p$ leads to

$$\sum_{j=1}^{n} \tau_{p,j} \sigma_{p,j}^{l} \hat{f}^{(j)}(-lh_j/h_p) = i \sum_{j=1}^{n} \tau_{p,j} \sigma_{p,j}^{l} \hat{g}^{(j)}(-lh_j/h_p), \quad p = 1, \ldots, n, \tag{46}$$

where $\sigma_{p,j} := e^{i(m_j - m_p)/h_p}$ and $\tau_{p,j} := (h_j/h_p) e^{-i(\delta_j - \delta_p)}$. Thus, (46) is a system of n equations for the n unknown functions $\hat{f}^{(j)}$. We note that $\sigma_{p,p} = 1$, and that for $j \neq p$, $|\sigma_{p,j}| < 1$ (by convexity) so the coefficient of $\hat{f}^{(j)}$ is exponentially small as $l \to \infty$. Also, for $j = p$ and $l \in \mathbb{Z}$ the numbers $\hat{f}^{(j)}(l)$ are simply the coefficients in the Fourier series for $f^{(j)}$. Since this function is real, the coefficients for $l \leqslant 0$ [i.e., $k \in \ell_j$ as needed for (13)] are related to those for $l \geqslant 0$ [i.e., $k \in \hat{\ell}_j$ as determined by (46)] via $\hat{f}^{(j)}(-l) = \overline{\hat{f}^{(j)}(l)}$.

[3] This argument assumes the domain D is convex and contains the origin. It can be made independent of the location of the origin by scaling by the coefficient of $\hat{f}^{(p)}$ as in (46).

4.3. Continuity conditions

Up to this point the only restriction imposed on the solution is that q_z must be integrable on each side. Near a vertex z_j the behavior of the solution depends on the interior angle $\omega_j := \pi - (\alpha_j - \alpha_{j-1}) \in (0, \pi)$ and the boundary conditions as determined by β_j and β_{j-1} [cf. (15)]. Without loss of generality we can assume that $0 \leqslant \beta_j - \beta_{j-1} < \pi$. It can be shown that:

- If $\omega_j + \beta_j - \beta_{j-1} < \pi$ then the problem is *regular* (q_z is bounded) near z_j.
- If $\omega_j + \beta_j - \beta_{j-1} > \pi$ then the problem is *singular* (q_z may be unbounded) near z_j.

In the borderline case $\omega_j + \beta_j - \beta_{j-1} = \pi$ we have $\delta_j = \delta_{j-1}$, which says the same component of the derivative (e.g., q_x) is specified along sides S_j and S_{j-1}. For simplicity, we exclude both the borderline and singular cases here. Then the endpoint values of the unknowns can be determined from the continuity of q_z, and this information can be used to overcome the degeneracy present in (46) when $l = 0$.

Specifically, at any vertex z_j we have two representations of $q_z(z_j)$, namely, $q_z^{(j-1)}(z_j)$ and $q_z^{(j)}(z_j)$. Requiring that these representations match and using (17) leads to the condition

$$e^{-i\delta_j}[g^{(j)}(-\pi) + i f^{(j)}(-\pi)] = e^{-i\delta_{j-1}}\left[g^{(j-1)}(\pi) + i f^{(j-1)}(\pi)\right]. \tag{47}$$

The real and imaginary parts of this equation yield two equations for the unknown values $f^{(j-1)}(\pi)$ and $f^{(j)}(-\pi)$. Solving these equations we find

$$f^{(j-1)}(\pi) = \frac{\cos(\delta_j - \delta_{j-1}) g^{(j-1)}(\pi) - g^{(j)}(-\pi)}{\sin(\delta_j - \delta_{j-1})} \tag{48}$$

and

$$f^{(j)}(-\pi) = \frac{g^{(j-1)}(\pi) - \cos(\delta_j - \delta_{j-1}) g^{(j)}(-\pi)}{\sin(\delta_j - \delta_{j-1})}. \tag{49}$$

Knowing these endpoint values, we can set

$$f^{(j)} = f_*^{(j)} + F^{(j)}, \tag{50}$$

where the *linear* functions $f_*^{(j)}$ are chosen so that $F^{(j)}(-\pi) = F^{(j)}(\pi) = 0$ for each $j = 1, \ldots, n$. Using (50) and an analogous substitution for $g^{(j)}$ reduces (46) to

$$\sum_{j=1}^{n} \tau_{p,j} \sigma_{p,j}^{l} \hat{F}^{(j)}(-l h_j/h_p) = i \sum_{j=1}^{n} \tau_{p,j} \sigma_{p,j}^{l} \left[\hat{G}^{(j)}(-l h_j/h_p) + \hat{H}^{(j)}(-l h_j/h_p)\right] \tag{51}$$

for $p = 1, \ldots, n$, where $\hat{H}^{(j)}(k)$ is the Fourier transform of the linear function

$$H^{(j)}(s) = g_*^{(j)}(s) + i f_*^{(j)}(s). \tag{52}$$

4.4. Numerical implementation

The global condition in form (51) consists of n equations for the n unknown functions $F^{(j)}$, $j = 1, \ldots, n$ (more specifically, their Fourier transforms), which are coupled through the argument

$k = -lh_j/h_p$. Note that each of the functions $F^{(j)}$ and $G^{(j)}$ is in $C_0[-\pi,\pi] = \{f \in C[-\pi,\pi] : f(-\pi) = f(\pi) = 0\}$, and that up to this point the formulation is continuous in k (or l). To discretize the problem, for each side S_j we choose a basis $\{\varphi_r^{(j)}(s)\}_{r=1}^{N_j}$ for a subspace S_j of $C_0[-\pi,\pi]$ (with dimension N_j even), and approximate $F^{(j)}(s)$ and $G^{(j)}(s)$ by

$$F_{N_j}^{(j)}(s) = \sum_{r=1}^{N_j} c_r^{(j)} \varphi_r^{(j)}(s) \tag{53}$$

and

$$G_{N_j}^{(j)}(s) = \sum_{r=1}^{N_j} d_r^{(j)} \varphi_r^{(j)}(s), \tag{54}$$

respectively. One example of such a basis is the *hat functions* defined on $[-\pi,\pi]$ with mesh spacing $2\pi/(N_j+1)$; this family of piecewise linear functions is popular in finite element methods. Another example is the sine basis

$$\varphi_r^{(j)}(s) = \sin\left[r\left(\frac{s+\pi}{2}\right)\right], \tag{55}$$

which we will use for the numerical results presented below.

The Fourier transforms of (53) and (54) are

$$\hat{F}_{N_j}^{(j)}(k) = \sum_{r=1}^{N_j} c_r^{(j)} \hat{\varphi}_r^{(j)}(k) \tag{56}$$

and

$$\hat{G}_{N_j}^{(j)}(k) = \sum_{r=1}^{N_j} d_r^{(j)} \hat{\varphi}_r^{(j)}(k), \tag{57}$$

respectively. Since $G^{(j)}$ is a known function, we can determine the coefficients $d_r^{(j)}$ by a collocation projection of (57) in k, i.e., we require $\hat{G}^{(j)}(k) = \hat{G}_{N_j}^{(j)}(k)$ for $k = l = 1,\ldots,M_j$ where $M_j = N_j/2$. This leads to the linear system

$$\sum_{r=1}^{N_j} d_r^{(j)} \hat{\varphi}_r^{(j)}(l) = \hat{G}^{(j)}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}ls} G^{(j)}(s)\,\mathrm{d}s, \tag{58}$$

the right-hand side of which can be computed by direct or numerical integration, or more efficiently by the FFT algorithm. Since the basis functions $\varphi_r^{(j)}$ are linearly independent, system (58) is uniquely solvable.

Likewise, we determine the coefficients $c_r^{(j)}$ by a collocation projection of the global condition (51) using approximations (56) and (57). The resulting discrete equations are

$$\sum_{j=1}^{n} \tau_{p,j}\sigma_{p,j}^{l} \sum_{r=1}^{N_j} c_r^{(j)} \hat{\varphi}_r^{(j)}(-lh_j/h_p)$$
$$= \mathrm{i}\sum_{j=1}^{n} \tau_{p,j}\sigma_{p,j}^{l}\left[\sum_{r=1}^{N_j} d_r^{(j)} \hat{\varphi}_r^{(j)}(-lh_j/h_p) + \hat{H}^{(j)}(-lh_j/h_p)\right], \tag{59}$$

which are applied for $l=1,\ldots,M_j$ and $p=1,\ldots,n$. Thus, the discrete global condition (59) consists of $M=\sum_{p=1}^{n} M_p$ complex equations for the $N=\sum_{j=1}^{n} N_j$ real coefficients $c_r^{(j)}$, where $N=2M$. The matrix form of this system is straightforward, although tedious due to the need to work with the real and imaginary parts separately; the details appear in Appendix A. The system could be solved by a natural block-iterative method (e.g., Jacobi or Gauss-Seidel); here for simplicity we solve it directly (using Matlab).

5. Numerical results

To illustrate the method, we apply it to the Laplace equation on a variety of convex polygonal domains with different boundary conditions. For concreteness, in each case we take the analytical solution to be

$$q(x,y)=\sinh(3x)\sin(3y) \tag{60}$$

and generate the corresponding boundary data analytically. To avoid unrepresentative results due to alignment with the coordinate axes, in each case the specified domain is rotated by the angle 0.2. The performance of the method is quantified by comparing the analytical solution f, composed piecewise of $f^{(j)}$ on sides S_j [cf. (50)], and the corresponding numerical solution f_N, composed piecewise of $f_*^{(j)}+F_{N_j}$ [cf. (53)]. For the figures we measure the maximum relative error in the norm

$$E_\infty := \frac{\|f-f_N\|_\infty}{\|f\|_\infty}, \quad \|f\|_\infty := \max_{1\leqslant j\leqslant n}\left\{\max_{-\pi\leqslant s\leqslant\pi}|f^{(j)}(s)|\right\} \tag{61}$$

with the max over s taken over a large number of discrete points. In all cases we use the same number of basis functions on each side of the domain.

First, we consider the solution for regular polygons. For $n=3,4,5,6$, and 8 we construct a domain as a regular n-gon centered at the origin with $z=1$ the midpoint of one side. Figs. 4 and 5 show the

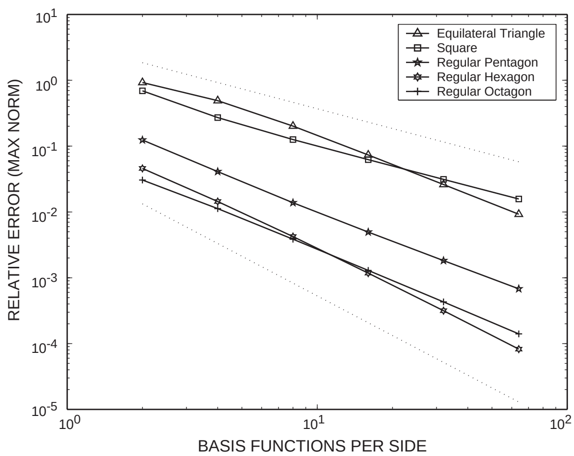

Fig. 4. Relative errors on regular polygons with Dirichlet boundary conditions.

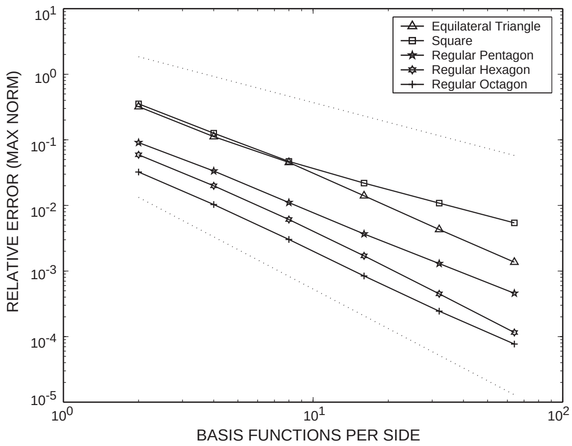

Fig. 5. Relative errors on regular polygons with Neumann boundary conditions.

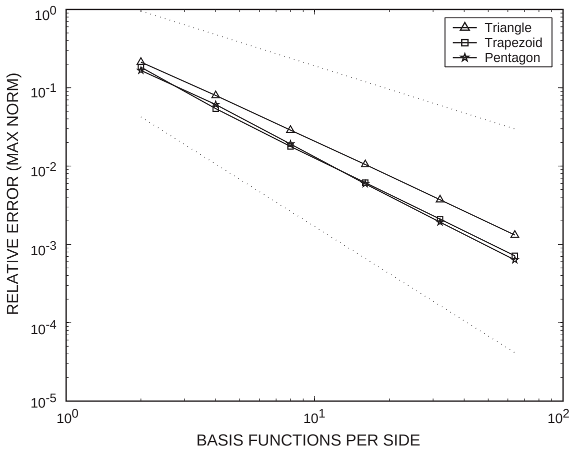

Fig. 6. Relative errors on general polygons with mixed boundary conditions.

corresponding errors for Dirichlet ($\beta_j = 0$) and Neumann ($\beta_j = \pi/2$) boundary conditions, respectively. The dotted lines in each figure show slopes for convergence of order 1 and 2. It is evident that the method converges in each case, with order of convergence between 1 and 2. These results were computed with the sine basis (55); with a basis of hat functions the corresponding errors (not shown) are similar but larger by at least a factor of two.

Likewise, Fig. 6 shows corresponding results for three more general polygons with mixed boundary conditions ($\beta_j = \pi/3$):

- A triangle with vertices $(0,0)$, $(0,2)$, and $(1,0)$.
- A trapezoid with vertices $(0,0)$, $(0,1)$, $(1,2)$, and $(1,0)$.
- A pentagon with vertices $(0,0)$, $(0,1)$, $(1,2)$, $(2,1)$, and $(1,0)$.

Once again, the numerical solution converges in each case.

6. Conclusions and remarks

A new numerical method for solving linear elliptic PDEs with constant coefficients has been introduced. Proposition 2 of Section 3 reduces the solution of the Laplace equation on an arbitrary convex bounded polygon to the solution of the global condition, which couples known and unknown components of the derivative on the boundary. The numerical method introduced in Section 4 provides an approximate solution of the global condition. Results presented in Section 5 demonstrate the convergence of the method; however, a rigorous proof of convergence remains open.

An advantage of this new numerical method is that it is formulated only in terms of the *boundary values*. This should be particularly useful if one is interested only in the "missing" boundary values, as opposed to $q(x,y)$ in the interior of the polygon. Examples of such cases include the analysis of potential fields [5] and the determination of the drag force in creeping (Stokes) flow [7].

We note that the method introduced here can be extended in various ways:

(1) *Problems with singularities at the vertices*: Recall from Section 4.3 that if $\omega_j+\beta_j-\beta_{j-1}\geqslant\pi$ then the problem is singular near z_j. In this case the corner values $q^{(j)}(\pm\pi)$ *cannot* be computed using the continuity conditions (47), but must be coupled with the unknown coefficients $c_r^{(j)}$.

(2) *Other boundary conditions*: We can replace the mixed boundary conditions (15) with the more general Poincaré boundary conditions

$$\cos(\beta_j)q_s^{(j)}+\sin(\beta_j)q_n^{(j)}+\gamma_j q^{(j)}=g^{(j)},\quad z\in S_j. \tag{62}$$

If $\gamma_j\neq 0$ we can assume without loss of generality that $\sin(\beta_j)\neq 0$; otherwise $q^{(j)}$ and hence $q_s^{(j)}$ is known, reducing to a special case of (15). Solving (62) for $q_n^{(j)}$, substituting the result into (10) to obtain q_z, substituting that into (9) and integrating by parts yields

$$\rho_j(k)=\frac{\mathrm{i}\pi h_j \mathrm{e}^{-\mathrm{i}\delta_j}\mathrm{e}^{-\mathrm{i}km_j}}{\sin(\beta_j)}\left\{\mathrm{e}^{-\mathrm{i}\beta_j}\hat{g}^{(j)}(kh_j)-(\gamma_j\mathrm{e}^{-\mathrm{i}\beta_j}+\mathrm{i}k\mathrm{e}^{\mathrm{i}\alpha_j})\hat{q}^{(j)}(kh_j)\right.$$
$$\left.-\frac{1}{2\pi|h_j|}\left[\mathrm{e}^{-\mathrm{i}kh_j\pi}q^{(j)}(\pi)-\mathrm{e}^{\mathrm{i}kh_j\pi}q^{(j)}(-\pi)\right]\right\}, \tag{63}$$

which replaces (44) in the development of the method. The global condition (8) can then be solved in a manner similar to that outlined in Section 4, with two differences. First, the discrete expansion must now be for the variable $q^{(j)}$, rather than a derivative of $q^{(j)}$. Second, the corner values $q^{(j)}(\pm\pi)$ in (63) cannot be eliminated in advance, but must be coupled with the unknown coefficients of the solution as in point 1 above.

(3) *Other linear elliptic PDEs*: The results obtained here can be generalized to other linear elliptic equations. For example, for the modified Helmholtz equation (1) with $\alpha=-\beta^2$, $\rho_j(k)$ is given by [cf. (5)]

$$\rho_j(k)=\int_{S_j}\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\beta^2\bar{z}/k}\left[q_z\,\mathrm{d}z+\frac{\mathrm{i}\beta^2}{k}\,q\,\mathrm{d}\bar{z}\right],\quad k\in\mathbb{C}, \tag{64}$$

where q satisfies Eq. (1) with $\alpha=-\beta^2$. In particular, the analysis of the global condition associated with other elliptic PDEs is conceptually similar to the analysis of the global relation (8). In this sense the application to the Laplace equation is the simplest but still a generic example of the application of a new approach to linear elliptic PDEs.

(4) *Forced problems*: Consider for example the Poisson problem

$$q_{z\bar{z}} = F(z,\bar{z}), \quad z \in D. \tag{65}$$

In this case the global condition (8) is replaced by

$$\sum_{j=1}^{n} \rho_j(k) = 2\mathrm{i} \int\int_D \mathrm{e}^{-\mathrm{i}kz} F(z,\bar{z})\,\mathrm{d}x\,\mathrm{d}y, \quad k \in \mathbb{C}. \tag{66}$$

The right-hand side of this equation is known, thus its analysis is similar to the analysis of Eq. (8). However, computing this right-hand side requires a double integration for each value of k, so this approach may be practical only if these integrals can be computed analytically.

Finally, several aspects of the numerical method introduced here deserve further investigation. In particular, in subsequent research we plan to address the convergence analysis, iterative solution methods, and comparison with other methods in terms of accuracy and efficiency.

Appendix A. Matrix form of the linear system

To write the linear system (59) in matrix form we treat the real and imaginary parts of the equations separately. To that end, for any $\lambda \in \mathbb{C}$ let

$$\hat{\mathbf{F}}_\lambda^{(p,j)} := [\mathscr{B}_1^{\mathrm{T}}(\hat{F}_{N_j}^{(j)}(-\lambda)), \mathscr{B}_1^{\mathrm{T}}(\hat{F}_{N_j}^{(j)}(-2\lambda)), \ldots, \mathscr{B}_1^{\mathrm{T}}(\hat{F}_{N_j}^{(j)}(-M_p\lambda))]^{\mathrm{T}} \tag{A.1}$$

in $\mathbb{R}^{2M_p}$, where $\mathscr{B}_1(z) := [\mathrm{Re}(z), \mathrm{Im}(z)]^{\mathrm{T}}$. Then applying (56) at $k = \lambda,\ 2\lambda, \ldots, M_p\lambda$ we can write the result in matrix form as

$$\hat{\mathbf{F}}_\lambda^{(p,j)} := \hat{P}_\lambda^{(p,j)} \mathbf{c}^{(j)}, \tag{A.2}$$

where $\mathbf{c}^{(j)} := \left[c_1^{(j)}, \ldots, c_{N_j}^{(j)}\right]^{\mathrm{T}}$ and

$$\hat{P}_\lambda^{(p,j)} := \begin{bmatrix} \mathscr{B}_1\left(\hat{\varphi}_1^{(j)}(-\lambda)\right) & \cdots & \mathscr{B}_1\left(\hat{\varphi}_{N_j}^{(j)}(-\lambda)\right) \\ \vdots & & \vdots \\ \mathscr{B}_1\left(\hat{\varphi}_1^{(j)}(-M_p\lambda)\right) & \cdots & \mathscr{B}_1\left(\hat{\varphi}_{N_j}^{(j)}(-M_p\lambda)\right) \end{bmatrix}. \tag{A.3}$$

For any $z \in \mathbb{C}$ we define the block-diagonal matrices

$$D_{M_p}(z) := \mathrm{diag}[\mathscr{B}_2(z), \mathscr{B}_2(z), \ldots, \mathscr{B}_2(z)], \tag{A.4}$$

$$E_{M_p}(z) := \mathrm{diag}[\mathscr{B}_2(z^1), \mathscr{B}_2(z^2), \ldots, \mathscr{B}_2(z^{M_p})] \tag{A.5}$$

in $\mathbb{R}^{2M_p\times 2M_p}$, where

$$\mathscr{B}_2(z) := \begin{bmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{bmatrix}. \tag{A.6}$$

Then if we write vector forms of $\hat{G}^{(j)}$ and $\hat{H}^{(j)}$ analogous to (A.1), the global condition (59) takes the matrix form

$$\sum_{j=1}^{n} D_{M_p}(\tau_{p,j})E_{M_p}(\sigma_{p,j})\hat{P}^{(p,j)}_{h_j/h_p}\mathbf{c}^{(j)}$$
$$= \sum_{j=1}^{n} D_{M_p}(\mathrm{i}\tau_{p,j})E_{M_p}(\sigma_{p,j})\left[\hat{P}^{(p,j)}_{h_j/h_p}\mathbf{d}^{(j)} + \hat{\mathbf{H}}^{(p,j)}_{h_j/h_p}\right] \tag{A.7}$$

for $p = 1,\ldots,n$, where $\mathbf{d}^{(j)} := \left[d_1^{(j)},\ldots,d_{N_j}^{(j)}\right]^{\mathrm{T}}$. We can write this linear system as

$$A\mathbf{c} = \mathbf{b}, \tag{A.8}$$

where $A \in \mathbb{R}^{N\times N}$ has blocks

$$A^{(p,j)} = D_{M_p}(\tau_{p,j})E_{M_p}(\sigma_{p,j})\hat{P}^{(p,j)}_{h_j/h_p}, \tag{A.9}$$

$\mathbf{c} = [(\mathbf{c}^{(1)})^{\mathrm{T}},\ldots,(\mathbf{c}^{(n)})^{\mathrm{T}}]^{\mathrm{T}}$ and similarly for $\mathbf{b}$, with $\mathbf{b}^{(p)}$ denoting the right-hand side of (A.7).

It should be noted that system (A.8) can also be written in the form

$$\hat{A}\hat{\mathbf{F}} = \mathbf{b}, \tag{A.10}$$

where the blocks comprising $\hat{A} \in \mathbb{R}^{N\times N}$ are

$$\hat{A}^{(p,j)} = A^{(p,j)}\left[\hat{P}^{(j)}\right]^{-1} \tag{A.11}$$

and $\hat{\mathbf{F}} = \left[(\hat{\mathbf{F}}^{(1)})^{\mathrm{T}},\ldots,(\hat{\mathbf{F}}^{(n)})^{\mathrm{T}}\right]^{\mathrm{T}}$ with $\hat{F}^{(j)} = \hat{F}_1^{(j,j)}$ and $\hat{P}^{(j)} = \hat{P}_1^{(j,j)}$. Indeed, (A.8) may be referred to as a "physical space" form of the global condition, since the unknown $\mathbf{c}$ gives the functions $F^{(j)}$ as functions of s via (53), and (A.10) may be referred to as a "spectral space" form, since the unknown $\hat{\mathbf{F}}$ gives the Fourier coefficients of those functions. The latter may appear preferable: the unknowns are (essentially) the unknown functions $\rho_j(k)$ needed to compute the solution of the Laplace equation [cf. (13)], and the diagonal blocks of $\hat{A}$ are identity matrices. Nevertheless, the former may be better in practice, since the condition number of the physical-space matrix A grows more slowly with N.

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