Basis Markov Partitions and Transition Matrices for Stochastic Systems

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We analyze dynamical systems subjected to an additive noise and their deterministic limit. In this work, we will introduce a notion by which a stochastic system has something like a Markov partition for deterministic systems. For a chosen class of the noise profiles the Frobenius-Perron operator associated to the noisy system is exactly represented by a stochastic transition matrix of a finite size $K$. This feature allows us to introduce for these stochastic systems a basis–Markov partition, defined herein, irrespectively of whether the deterministic system possesses a Markov partition or not. We show that in the deterministic limit, corresponding to $K \rightarrow \infty$, the sequence of invariant measures of the noisy systems tends, in the weak sense, to the invariant measure of the deterministic system. Thus by introducing a small additive noise one may approximate transition matrices and invariant measures of deterministic dynamical systems.

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I. INTRODUCTION

Markov partitions for deterministic dynamical systems serve a central role for determining their symbolic dynamics, [1–3] whose grammar is described by a finite sized transition matrix that generates a so-called sofic shift [4, 5]. The conditions for such a projection were defined by Bowen for Anosov hyperbolic systems [1, 3], and stated succinctly for interval maps as a partition whose elements are each a homeomorphism onto a finite union of its elements [2, 3]. We remark here that a defining property in both cases is that the set of characteristic functions defined over the elements of the Markov partition project the transfer operator exactly onto an operator of finite type; that is a matrix results whereas an infinite matrix would be expected for a non-Markov system. We argue here that this should be the defining property of any generalization of Markov partitions; that is, a set of basis functions which project the Frobenius-Perron operator exactly onto a finite-rank matrix with no residual.

First we recall the Frobenius-Perron operator for a deterministic transformation. Associated with a discrete dynamical system acting on initial conditions, $z \in M$ (say a manifold, $M \subset \mathbb{R}^n$),

\[ F : M \rightarrow M, \]
\[ x \mapsto F(x), \] (1)

is another dynamical system over $L^1(M)$, the space of densities of ensembles of initial conditions.

\[ P_F : L^1(M) \rightarrow L^1(M), \]
\[ \rho(x) \mapsto P_F[\rho(x)]. \] (2)

This Frobenius-Perron operator ($P_F$) is defined through a continuity equation [6],

\[ \int_{F^{-1}(B)} \rho(x)dx = \int_B P_F[\rho(x)]dx, \] (3)

for measurable sets $B \subset M$. Differentiation changes this operator equation to the commonly used form,

\[ P_F[\rho(x)] = \int_M \delta(x - F(y))\rho(y)dy, \] (4)

acting on probability density functions $\rho \in L^1(M)$. 

Now consider the stochastically perturbed dynamical system

\[ F_\nu : M \rightarrow M, \]
\[ x \mapsto F(x) + \xi, \]  

where \( \xi \) is an i.i.d. random variable with PDF \( \nu(z) \), which is applied once per each iteration. The random part \( \nu \) is assumed to be independent of state \( x \) which we tacitly assume to be relatively small, so that the deterministic part \( F \) has primary influence. The “stochastic Frobenius-Perron operator” has a similar form to the deterministic case [6],

\[ P_{F_\nu} [\rho(x)] = \int_M \nu(x - F(y)) \rho(y) dy, \]  

where the deterministic kernel, the delta function in Eq. (2), now becomes a stochastic kernel describing the PDF of the noise perturbation. In the case that the random map Eq. (5) arises from the usual continuous Langevin process, the infinitesimal generator of the Frobenius-Perron operator (FP–operator) for normal \( \nu \) corresponds to a general solution of a Fokker-Planck equation, [6]. The Frobenius-Perron operator formalism is particularly convenient in that it allows for an arbitrary noise distribution \( \nu \) to be incorporated in a direct and simple way. Within the formalism, we can also study multiplicative noise \((x \rightarrow \eta F(x), \) modeling parametric noise). The kernel-type integral transfer operator is, \( K(x, y) = \nu(x/F(y))/F(y) \) for \( x \in \mathbb{R}^+ \), which can then also be finitely approximated as described in the next section, and usefully re-ordered to canonical block reduced form. In more generality, the theory of random dynamical systems [7] clearly classifies those random systems which give rise to explicit transfer operators with corresponding infinitesimal generators, and there are well defined connections between the theories of random dynamical systems and of stochastic differential equations.

The main aim of this work is to investigate a class of stochastically perturbed dynamical systems for which the FP operator is represented by a finite stochastic transition matrix of size \( K \). Such dynamical systems will be called basis–Markov in analogy to deterministic dynamical systems possessing a Markov partition, for which the associated FP operator is finite. The deterministic limit of the stochastic system corresponds to the divergence of the matrix size. In this limit, \( K \rightarrow \infty \), the sequence of invariant measures of the stochastic systems acting in the \( K \)-dimensional Hilbert space converges, in the weak sense, to the invariant measure of the corresponding deterministic system.
The paper is organized as follows. The Ulam-Galerkin method of approximating the infinite dimensional FP operator and the concept of the Markov partition for a deterministic system are reviewed in sections II and III, respectively. In section IV we introduce the notion of basis-Markov stochastic systems, while in section V we analyze a particular example of random systems perturbed by an additive noise with cosine profile. The key result on convergence of the invariant measures for stochastic and deterministic systems is proved in section VI. A discussion of isospectral matrices used to describe the FP operator is relegated to the appendix.

II. ULAM-GALERKIN’S METHOD-APPROXIMATING THE INFINITE-DIMENSIONAL OPERATOR

A Galerkin’s method may be used to approximate the Frobenius-Perron operator by a Markov operator of finite rank. Formally, projection of the infinite dimensional linear space $L^1(M)$ results with discretely indexed basis functions $\{\phi_i(x)\}_{i=1}^{\infty} \subset L^1(M)$ onto a finite dimensional linear subspace generated by a subset of the basis functions [8],

$$\Delta_N = \text{span}(\{\phi_i(x)\}_{i=1}^{N}),$$

such that $\phi_i \in L^1(M) \ \forall i$. This projection,

$$p : L^1(M) \rightarrow \Delta_N,$$

is realized optimally by the Galerkin method in terms of the inner product, which we choose to be integration,

$$(f, g) \equiv \int_M f(x)g(x)dx, \forall f, g \in L^2(M).$$

Specifically, the infinite-dimensional “matrix” is approximated by the $N \times N$ matrix,

$$A_{i,j} = (P_{F_{\nu}}[\phi_i], \phi_j) = \int_M P_{F_{\nu}}[\phi_i(x)]\phi_j(x)dx, \ 1 \leq i, j \leq N.$$  

One approximates $\rho(x)$, through a finite linear combination of basis functions,

$$\rho(x) \simeq \sum_{i=1}^{N} d_i\phi_i(x).$$

The historically famous Ulam’s method [9] for deterministic dynamical systems is equivalent to the interpretation to find the fraction of the box $B_i$ which maps to $B_j$; the Ulam matrix
is equivalent to the Galerkin matrix by using Eq. (10) and choosing the basis functions to be the family of characteristic functions,

\[ \phi_i(x) = 1_{B_i}(x) = \begin{cases} 
1 & \text{if } x \in B_i \\
0 & \text{else.}
\end{cases} \] (12)

Specifically, we choose the ordered set of basis functions to be in terms of a nested refinement of boxes \( \{B_i\} \) covering \( M \). Though Galerkin’s and Ulam’s methods are formally equivalent in the deterministic case, we are of the opinion that the Galerkin description is a more natural description in the stochastic setting.

**III. MARKOV PARTITIONS OF DETERMINISTIC SYSTEMS, AND EXACT PROJECTION**

In this section, we discuss that a Markov partition is special for the Frobenius-Perron operator of a deterministic dynamical system, in that characteristic functions supported over those partition elements leads to an exact projection of the FP operator onto an operator of finite rank - a matrix.

For a one-dimensional transformation of the interval, a Markov partition is defined [2, 3],

**Definition:** A map of the interval \( f : [a, b] \to [a, b] \) is Markov if there is a finite partition \( \{I_i\} \) such that,

1. \( \cup_j I_j = [a, b] \) (covering property),
2. \( int(I_j) \cap int(I_k) = \emptyset \) if \( k \neq k \) (no overlap property),
3. \( f(I_j) = \cup_k I_{k_i} \), (a grid interval maps completely across a union of intervals without “dangling ends” property).

It is not hard to show that the set of characteristic functions forms a finite basis set of functions

\[ \{\phi_i(x)\} = \{1_{I_i}(x)\}_i, \] (13)

such that Galerkin projection Eq. (10) is exact onto an operator of finite rank, or a matrix \( A_{i,j} \). That is, Eq. (10) simplifies,

\[ A_{i,j} = (P_{\nu}[\phi_i], \phi_j) = \int_M P_{\nu}[\phi_i(x)]\phi_j(x)dx, \]
\[ \int_M \delta(x - F(y)) \phi_i(y) \phi_j(x) dy dx = \int_{I_i} \int_{I_j} \delta(x - F(y)) dy dx \quad 1 \leq i, j \leq N. \tag{14} \]

From the definition of the Markov partition, we see that a row of \(A_{i,j}\) accounts that \(P_{F_*} [\phi_i(x)]\) is a linear combination of \(\phi_j(x)\).

Similarly, there is a well defined notion of an Anosov diffeomorphisms with a Markov partition [1, 3, 10, 11], and so for such systems, it can be shown that characteristic functions supported over the corresponding Markov partition creates a basis set such that Eq. (10) results in an operator of finite rank.

We take these observations as motivation to make the following definition which is meant to generalize the notion of a Markov partition to stochastic systems:

**Definition:** Suppose a measure space, \(\{M, \mathcal{B}, \mu\}\), and a transformation \(F : M \to M\), then the transformation is **basis Markov** if there exists a finite set of basis functions \(\{\phi_i(x)\}_{i=1}^n : M \to [0, 1] \in L^1(M)\) such that the Frobenius-Perron operator is operationally closed within \(\Delta_n\), where \(\Delta_n = \text{span}(\{\phi_i(x)\}_{i=1}^n)\). That is, for any probability measure \(\rho\), its image \(P_F[\rho(x)]\) belongs to \(\Delta_n\).

**Remark 1:** If a transformation \(F\) is basis-set Markov, then if we perform Galerkin’s method, \(A_{i,j} = (P_{F_*} [\phi_i], \phi_j)_{M \times M}\), with that basis set, then it allows that for any initial density which can be written as a linear combination of these basis functions,

\[ \rho_0(x) = \sum_{i=1}^n c_i \phi_i(x), \tag{15} \]

or stated simply,

\[ \rho_0(x) \in \Delta_n, \tag{16} \]

then the action of the Frobenius-Perron operator on such initial densities, \(\rho_1(x) = P_{F_*} [\rho_0(x)]\), can be exactly represented by the following matrix-vector multiplication:

\[ \mathbf{c}' = A \cdot \mathbf{c}, \text{ where } \rho_1(x) = \sum_{i=1}^n c'_i \phi_i(x). \tag{17} \]

That is, the FP operator projects exactly to an operator of finite rank - a matrix.

Note that for a general finite set of functions, if we take a general linear combination of those functions and then apply the Frobenius-Perron operator, we do not expect the resulting density can be written as a (finite) linear combination of basis functions.
The following is a direct consequence of our definition of basis Markov in relationship to the usual definition of a Markov map, stating the sense in which basis Markov is a generalization:

**Remark 2:** Given a Markov map, then Eq. (14) implies that any Markov map, together with the characteristic functions supported over the partition elements, is basis Markov.

**IV. BASIS MARKOV STOCHASTIC SYSTEMS: A GENERAL CASE DUE TO SEPARABLE NOISE**

We analyze a dynamical system defined on an interval \( M = [0, 1] \) with both ends identified and subjected to a specific form of the additive noise,

\[
x' = f(x) + \xi.
\]  

(18)

To specify the special case of the stochastic dynamical system written in Eq. (5), the stochastic perturbation will be characterized by the probability \( P(x, y) \) of a transition form point \( x \) to \( y \) induced by noise. Describing the dynamics in terms of a probability density \( \rho(x) \) its one-step evolution is governed by the stochastic *Frobenius-Perron (FP) operator,*

\[
\rho'(y) = P_f(\rho(y)) = \int P(f(x), y) \rho(x) dx.
\]  

(19)

We will denote this stochastic Frobenius-Perron operator by the symbol \( P_f \), in all that follows. The operator \( P_f \) acts on every probability measure defined on \( M \) and in general, it cannot be represented by a finite matrix. However, in the sequel we shall analyze a certain class of noise profiles for which such a representation is possible.

We assume that the transition probability \( P(x, y) \) satisfies the following properties \([12, 13]\):

\begin{align*}
\text{a)} & \quad P(x, y) \equiv P(x - y) = P(\xi), \\
\text{b)} & \quad P(x, y) \equiv P(x \text{ mod } 1, y \text{ mod } 1), \\
\text{c)} & \quad P(x, y) = \sum_{m,n=0}^{N} A_{mn} u_n(x) v_m(y),
\end{align*}

(20)

for \( x, y \in \mathbb{R} \) and an arbitrary finite \( N \). Property a) assures that the distribution of the random variable \( \xi \) does not depend on the position \( x \), while the periodicity condition is
provided in b). A noise profile fulfilling the latter property c) is called separable (decomposable), and it allows us to represent the dynamics of an arbitrary the system with such a noise in a finite dimensional Hilbert space. Here $A = (A_{mn})_{m,n=0,\ldots,N}$ is a yet undetermined real matrix of expansion coefficients. Note that $A$ characterizes the noise and does not depend on the deterministic dynamics $f$. We assume that the functions $u_n; \ n = 0, \ldots, N$ and $v_m; \ m = 0, \ldots, N$ are continuous in $X = [0,1)$ and linearly independent, so we can express $f \equiv 1$ as their linear combinations. Both sets of functions span bases in an $N + 1$ Hilbert space. Their orthogonality is not required.

This name "separable noise" is concocted in an analogy to separable states in quantum mechanics and separable probability distributions, since such a property was called $N + 1$-separability by Tucci [14]. Making use of this crucial feature of the noise profile we may expand the kernel of the Frobenius–Perron operator (19),

$$
\rho'(y) = P_f(\rho(y)) = \int_0^1 \sum_{m,n=0}^N A_{mn} u_n(f(x)) v_m(y) \rho(x) dx
$$

for $y \in X$, where,

$$
\tilde{v}_n = \sum_{m=0}^N A_{mn} v_m.
$$

Expanding $\rho'$ in an analogous way we will describe it by the vector $\vec{q}' = \{q'_0, \ldots, q'_N\}$. Assuming that a given density $\rho(x)$ belongs to this space, we can be expand it in this basis,

$$
\rho(x) = \sum_{m=0}^N q_m \tilde{v}_m(x).
$$

Expanding $\rho'$ in an analogous way we will describe it by the vector $\vec{q}' = \{q'_0, \ldots, q'_N\}$. Let $B$ denotes a matrix of integrals,

$$
B_{nm} = \int_0^1 u_n(f(x)) v_m(x) dx,
$$

where $n, m = 0, \ldots, N$. Observe that $B$ depends directly on the system $f$ and on the noise via the basis functions $u$ and $v$. Making use of this matrix, the one–step dynamics (23) may
be rewritten in a matrix form

\[ q'_n = \sum_{m=0}^{N} D_{nm} q_m, \quad \text{where} \quad D = BA \]  \tag{26}

and \( A \) is implied by (20). In this way we have arrived at a representation of the Frobenious–Perron operator \( P_f \) by a matrix \( D \) of size \( N + 1 \times N + 1 \), the elements of which read,

\[ D_{nm} = \int_0^1 u_n(f(x)) \tilde{v}_m(x) \, dx, \quad n, m = 0, \ldots, N. \]  \tag{27}

Although the probability is conserved under the action of \( P_f \), the matrix \( D \) need not be stochastic. This is due to the fact that the functions \( \{ \tilde{v}_m(x) \} \) forming the expansion basis in (24) were not normalized. We shall then compute their norms,

\[ \tau_m = \int_0^1 \tilde{v}_m(y) \, dy = \sum_{n=0}^{N} A_{mn} b_n \]  \tag{28}

where,

\[ b_n = \int_0^1 v_n(y) \, dy. \]  \tag{29}

Let \( K \leq N + 1 \) denote the number of non-zero components of the vector \( \vec{\tau} \) and let \( k = 1, \ldots, K \) runs over all indexes \( n \in 0, \ldots N + 1 \), for which \( \tau_k \neq 0 \). Then the rescaled vectors,

\[ V_k(y) := \tilde{v}_k(y)/\tau_k, \]  \tag{30}

are normalized,

\[ \int_0^1 V_k(y) \, dy = 1. \]  \tag{31}

The normalization condition \( \int_0^1 \rho(x) \, dx = 1 \) implies

\[ \int_0^1 \sum_{m=0}^{N} q_l \tilde{v}_m(x) \, dx = \sum_{m=0}^{N} q_m \tau_m = \sum_{k=1}^{K} q_k \tau_k = 1 \]  \tag{32}

The same is true for the transformed density,

\[ \sum_k q'_k \tau_k = 1. \]  \tag{33}

Hence this scalar product is preserved during the time evolution. Making use of the rescaled coefficients

\[ c_k := q_k \tau_k, \]  \tag{34}
Several values of $N$ are shown, with decreasing standard deviation with increasing $N$.

The dynamics (26) reads

$$c'_k = q'_k \tau_k = \sum_j D_{kj} q_j \tau_k = \sum_j D_{kj} \frac{\tau_k}{\tau_j} q_j \tau_j =: \sum_j T_{kj} c_j. \quad (35)$$

The coefficients $c_k$ sum to unity, so the transition matrix

$$T_{kj} \equiv D_{kj} \frac{\tau_k}{\tau_j} = \sum_{i'i'} D_{kj} \frac{A_{ki} \tau_i}{A_{ji'} \tau_{i'}} \quad (36)$$

is stochastic. In the above equation, all indices run from 1 to $K$ and the coefficients $\tau_k$ are non-zero by construction. Hence the dynamics (26) effectively takes place in an $K$-dimensional Hilbert space, and the Frobenious–Perron operator $P_f$ is represented by a stochastic matrix $T$ of size $K \times K$. The dimensionality $K \leq N + 1$ is determined by the parameter $N$ and the choice of the basis functions $\{v_i(x)\}$ entering (20).

V. A SPECIAL CASE: COSINE NOISE

We will now discuss a particularly simple case of the separable noise described above, introduced in [12]. Let,

$$\mathcal{P}_N(\xi) = C_N \cos^N(\pi \xi), \quad (37)$$
where \( N \) is even \((N = 0, 2, \ldots)\) and with the normalization constant,

\[
C_N = \sqrt{\frac{\pi}{2}} \frac{\Gamma[N/2 + 1]}{\Gamma[(N + 1)/2]}.
\] (38)

See Fig. 1, in which we can see the decreasing standard deviation with respect to increasing \( N \), and it can be seen that this type of noise reminds of a normal distribution, but of compact support.

The parameter \( N \) controls the strength of the noise measured by its variance

\[
\sigma^2 = \frac{1}{2\pi^2} \Psi'(\frac{N}{2} + 1) = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{m=1}^{N/1} \frac{1}{m^2},
\] (39)

where \( \Psi' \) stands for the derivative of the digamma function.

For the expansion (20) we use basis functions,

\[
u_n(y) = \cos^n(\pi y) \sin^{N-n}(\pi y),
\] (40)

where \( x \in X \) and \( m, n = 0, \ldots, N \). Expanding cosine as a sum to the \( N \)-th power in (37) we find that the \((N + 1) \times (N + 1)\) matrix \( A \) defined by (20) is diagonal,

\[
A_{mn} = a_m \delta_{mn}, \text{ with } a_m = C_N \binom{N}{m}.
\] (41)

Integrating trigonometric functions we find the coefficients,

\[
b_m = \int_0^1 \sin^m(\pi x) \cos^{N-m}(\pi x) dx = \frac{2}{\pi N} \frac{\Gamma[(m+1)/2] \Gamma[(N-m+1)/2]}{\Gamma(N/2)},
\] (42)

and,

\[
\tau_m = a_m b_m,
\] (43)

which are non-zero only for even values of \( m \). Hence the size \( K \times K \) of the transition matrix reads,

\[
K = N/2 + 1,
\] (44)

and the expression (36) takes the form

\[
T_{kj} = D_{mn} \frac{a_{mn} b_m}{a_n b_n} \quad \text{where} \quad k, j = 1, \ldots, K; \quad m = 2(k-1), \quad n = 2(j-1).
\] (45)

We find in the cosine noise Eq. (37) and with basis Eqs. (40), that the transition kernel reminds of a fuzzy but periodically repeated version of the map. See Fig. 2. However, the
Frobenius-Perron operator embeds to a transition matrix $T$, which “appears” roughly as a different form of the original map. See Fig. 3. However, with zero-noise, an Ulam transition matrix approximating the Frobenius-Perron operator would appear as the deterministic map. For this reason, we define the limit of $T$ matrices as $K \to \infty$ to be a singular limit of the Frobenius Perron operators, and associated the associated transformations.

There is an interesting correspondence between the spectrum of eigenvalues of the two matrices $D$ and $T$. Since $T$ is stochastic its largest eigenvalue is equal to unity. Moreover, it is the only eigenvalue with modulus one, which follows from the fact that the kernel $P(x, y)$ vanish only for $x - y = 1/2 \pmod{1}$, and the two-step probability function is everywhere positive,

$$\int P(x, z)P(z, y)dy > 0, \text{ for } x, y \in X.$$  \hspace{1cm} (46)

See [6], Th. 5.7.4. A particularly useful consequence and simplification is that the eigenstate corresponding to the largest eigenvalue of the matrix represents the invariant density of the system, $\rho_* = P_f(\rho_*)$; this can be easily found numerically by diagonalizing $T$.

All of the other eigenvalues are included inside the unit circle and their moduli $|\lambda_i|$ characterize the decay rates. It is worth emphasizing that the spectra of both matrix representations of the FP-operator - by matrices $D$ of size $(N + 1) \times (N + 1)$ used in [12, 13, 15] and the stochastic $T$ matrices of size $(N/2 + 1) \times (N/2 + 1)$, developed here, coincide up to the additional $N/2$ eigenvalues which are equal to zero – see the Appendix for details.

For concreteness let us discuss an exemplary 1-D dynamical system, a tent map:

$$f(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2(1 - x) & \text{if } 1/2 \leq x \leq 1. \end{cases} \hspace{1cm} (47)$$

Simple integration allows us to obtain analytic form of the transition matrix $T^{(N)}$ for the tent map (47) perturbed by additive noise characterized by small values of $N$,

$$T^{(2)} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad T^{(4)} = \frac{1}{24} \begin{bmatrix} 11 & 3 & 11 \\ 6 & 6 & 6 \\ 7 & 15 & 7 \end{bmatrix}, \quad T^{(6)} = \frac{1}{320} \begin{bmatrix} 145 & 25 & 25 & 145 \\ 69 & 45 & 45 & 69 \\ 51 & 75 & 75 & 51 \\ 55 & 175 & 175 & 55 \end{bmatrix}. \hspace{1cm} (48)$$

In the simplest case $N = 2$ the transition matrix is bistochastic, but it is not so for larger $N$. However, for this system, the matrix $T^{(N)}$ is of rank one for arbitrary value of the noise.
FIG. 2: The transition kernel $P_N(f(x), y)$ for the logistic map $f(x) = 4x(1 - x)$, with $N = 20$ and with cosine noise due to $N = 20$; compare to Fig. 1. Note the periodicity of $x$ of period-1.

parameter $N$. The spectrum of $T$ contains one eigenvalue equal to unity and all others equal to zero. This implies that every initial density is projected onto an invariant density already after the first iteration of the map. This is not the case for other dynamical systems $f$ including the logistic map $f_r(x) = rx(1 - x)$, for which the spectrum contains several resonances - eigenvalues of moduli smaller than one, which describe the decaying modes of the system [12].

VI. APPROXIMATION BY BASIS MARKOV MAPS

While not all maps and noise profiles allow for the map to be basis Markov, in this section we will show that a non basis Markov map may be weakly well approximated by basis Markov maps. In this sense, the finite approximations offered by basis Markov maps can be thought of as a good description of the general behavior, since the invariant measures of the finite approximations due to the basis Markov maps have weak-* limits to the invariant measures of the general maps.
FIG. 3: The stochastic matrix $T_{150}$ shown, from Eq. (36), exactly represents the stochastic Frobenius-Perron operator of the stochastic tent map Eq. (47) with trig noise Eq. (37), and basis set Eq. (40), using $N = 150$. Note that $T^{(150)}$ is a matrix of size $N/2 + 1 = 150/2 + 1 = 76$ square. Compare to the matrices in Eq. (48), of smaller $N$.

Considering the transition probabilities in Eqs. (19)-(20), we now write $\mathcal{P}_N(x, y)$ to denote the subindex $N$ to describe the finite number of terms sufficient to describe the probability in assumption (20)c. We require the following assumptions about the transition probabilities $\mathcal{P}_N(\cdot, \cdot)$:

1. $\mathcal{P}_N(\cdot, \cdot)$ is measurable as a function of two variables;

2. For every $x$ we have, $\int_0^1 \mathcal{P}_N(x, y)dy = 1$.

3. For every $y \in X$ we have

   $$\int_X \mathcal{P}_N(x, y)dx = 1.$$ 

4. Let $B(x, r) = \{y : |x - y| < r\}$ and,

   $$p_N(x, r) = \int_{X \setminus B(x, r)} \mathcal{P}_N(x, y)dy.$$  

   (49)
Then, for any \( r > 0 \),

\[
p_N(r) = \sup_{x \in X} p_N(x, r) \to 0, \quad \text{as} \quad N \to +\infty.
\]

Assumptions 1-3 are typical for probability measures, while assumption 4 is also rather mild, and it is easy to check that all four assumptions are satisfied by the cosine noise Eq. (37).

Under these assumptions, the following is true:

**Proposition:** For any \( \rho \in L^1(X) \) we have

\[
\int_X \rho(x) \mathcal{P}_N(x, y) dx \to \rho(y), \quad \text{as} \quad N \to \infty
\]

in \( L^1(X) \).

**Proof:** Let us assume that \( \rho \) is uniformly continuous and let us fix an \( \varepsilon > 0 \). We can find an \( r > 0 \) such that \(|\rho(x) - \rho(y)| < \varepsilon\) whenever \(|x - y| < r\). We have

\[
\int_X |\rho(y) - \int_X \rho(x) \mathcal{P}_N(x, y) dx| dy = \int_X \int_X |\rho(y) - \rho(x)| \mathcal{P}_N(x, y) dx dy
\]

(assumption (3)) = \( \int_X \int_X |\rho(y) - \rho(x)| \mathcal{P}_N(x, y) dx dy \leq \int_X \int_X |\rho(y) - \rho(x)| \mathcal{P}_N(x, y) dx dy \)

\[
= \int \int_{\{(x,y):|x-y|<r\}} |\rho(y) - \rho(x)| \mathcal{P}_N(x, y) dx dy + \int \int_{\{(x,y):|x-y|\geq r\}} |\rho(y) - \rho(x)| \mathcal{P}_N(x, y) dx dy
\]

\[
\leq \varepsilon \cdot 1 + 2 \cdot (\max_X |\rho|) \cdot p_N(r).
\]

The last estimate can be made arbitrarily small in view of assumption (4).

Once the convergence is proven for uniformly continuous functions, the proof of convergence for general \( L^1(X) \) functions is standard:

Since the natural norm of the operator \( \mathcal{P}_N(\rho)(y) = \int_X \rho(x) \mathcal{P}_N(x, y) dx \) in \( L^1(X) \) is 1, we have

\[
\|\rho - \mathcal{P}_N(\rho)\| \leq \|\rho - \rho_c\| + \|\rho_c - \mathcal{P}_N(\rho_c)\| + \|\mathcal{P}_N(\rho_c) - \mathcal{P}_N(\rho)\|
\]

\[
\leq \|\rho - \rho_c\| + \|\rho_c - \mathcal{P}_N(\rho_c)\| + \|\mathcal{P}_N\| \|\rho_c - \rho\|
\]

\[
\leq 2\|\rho - \rho_c\| + \|\rho_c - \mathcal{P}_N(\rho_c)\|,
\]

(52)
where $\rho_c$ is a uniformly continuous approximation of $\rho$. □

We will say that the transformation $f : [0,1] \to [0,1]$ preserves continuity iff for any continuous function $\rho$ the composition $\rho \circ f$ is also continuous. Obviously any continuous transformation $f$ preserves continuity. There are discontinuous maps which also preserve continuity, e.g., $f(x) = kx - \text{Int}(kx)$, for integer $k \geq 2$. We now confirm the following result, stated in the language of our current problem,

**Theorem: 1.** Let the transformation $f$ be continuity preserving. Under the assumptions (1), (2), and (4), it follows that if $\mu_N$ is an invariant measure of the stochastic perturbation of transformation $f$ defined by the transition probability $P_N$, then every weak-* limit point of the set $\{\mu_N : N \geq 1\}$ is an $f$-invariant measure.

**Proof:** This theorem can be proved following the ideas from, R.Z. Khasminskii, [16]. Let us assume that $\mu_N \to \mu$ weakly as $N \to +\infty$. We want to show that $\mu$ is $f$-invariant. To this end it is enough to show that

$$\int_X \rho d\mu = \int_X \rho(f) d\mu,$$

for any continuous function $\rho$. The stochastic perturbation of $f$ defined by transition probability $P_N$ acts on continuous functions as a composition with $f$ followed by application of the operator $P^*P_N$, defined as follows

$$(P^*P_N\rho)(x) = \int_X \rho(y)P_N(x,y)dy.$$

This operator is conjugated to the operator $P_{P_N}$ defined in the proof of the previous theorem. $P^*P_N$ acts on functions, while $P_{P_N}$ acts on functions understood as densities. Thus,

$$\int_X \rho d\mu_N = \int_X P_{P_N}^*(\rho(f)) d\mu_N,$$

for any continuous function $\rho$. Using assumption (4) we obtain

$$|\rho(x) - (P_{P_N}^*\rho)(x)|$$

$$= |\int_X \rho(x)P_N(x,y)dy - \int_X \rho(y)P_N(x,y)dy|$$

$$\leq \int_{\{y : |y-x| < r\}} |\rho(x) - \rho(y)|P_N(x,y)dy + \int_{\{y : |y-x| \geq r\}} |\rho(x) - \rho(y)|P_N(x,y)dy$$

$$\leq \varepsilon + 2 \cdot (\max_X |\rho|) \cdot p_N(r),$$

where $\varepsilon$ and $r$ are as in the proof of the previous theorem. This shows that $P_{P_N}^*(\rho)$ converges uniformly to $\rho$ for any continuous function $\rho$ as $N \to +\infty$. 
We have
\[ |\int_{X} \rho d\mu - \int_{X} \rho(f) d\mu| \leq |\int_{X} \rho d\mu - \int_{X} \rho d\mu_N| + |\int_{X} \rho d\mu_N - \int \mathcal{P}_{\mathcal{F}_{N}}(\rho(f)) d\mu_N| + |\int \mathcal{P}_{\mathcal{F}_{N}}(\rho(f)) d\mu_N - \int_{X} \rho(f) d\mu_N| + |\int_{X} \rho(f) d\mu_N - \int_{X} \rho(f) d\mu| \]

The first and the last differences converge to 0 since \( \mu_N \) converge \( \ast \)-weakly to \( \mu \). The second difference is 0 by the definition of \( \mu_N \). The third difference converges to 0 by the uniform convergence established just before. This proves Theorem 1. \( \Box \)

In this way we have established a relation between a sequence of noisy systems \( f_N \) and the deterministic dynamical system \( f \). A stochastic system (18) with the noise profile (37) for a fixed noise parameter \( N \) is described by a stochastic matrix \( T^{(N)} \) of size \( K = N/2 + 1 \) and acts in the Hilbert space \( \mathcal{H}_K \).

We have shown that the sequence of stochastic matrices \( T^{(N)} \), corresponds to the dynamical system \( f \), in a sense that the sequence \( \mu_N \) of the invariant measures of \( T^{(N)} \) converge weakly to the \( f \)-invariant measure \( \mu \) in the deterministic limit \( N \to \infty \). Furthermore, for any initial density \( \rho \) the sequence of vectors \( \rho'_N \) transformed by \( f_N \) converges weakly to the density transformed by the Frobenious–Perron operator associated with \( f \). Observe that the above property holds not only for one-dimensional systems, but also dynamical system \( f \) in higher dimensional measure spaces.

VII. CONCLUDING REMARKS

In this work we have introduced the concept of basis–Markov stochastic systems, for which the associated Frobenius–Perron operator is finite. This property resembles the class of deterministic systems with a Markov partition. However, the Markov partition is characteristic to a very special class of deterministic systems, while the basis–Markov property is related to the kind of stochastic perturbation. It holds for any deterministic system \( f \), subjected to an additive noise with a profile satisfying the separability condition (20). In this way such a random dynamical system can be described by a stochastic transition matrix of a finite size \( K \), which diverges in the deterministic limit.

We have shown an intimate relationship between the sequence of stochastic matrices which act in the space of \( K \)-point probability distributions and the FP operator \( P_f \) of
the deterministic system, which acts in the infinite dimensional space: In the deterministic limit $K \to \infty$ the invariant densities of stochastic matrices converge in a weak sense to the invariant measure of the deterministic system $f$. Thus constructing the transition matrices $T$ and decreasing the noise strength (and increasing the dimensionality $K$) one may construct arbitrary approximations of the FP operator $P_f$.

Note that the described method of finding an approximate invariant density of a deterministic system by applying a weak noise is not restricted to one dimensional systems. On the contrary, the entire construction can be directly applied for a general case of multi-dimensional dynamical systems. In particular, the definition (20)c of separable noise profiles works for the case of an $L$-dimensional systems, provided the variables $x$ and $y$ represent vectors with $L$ components each.

If the dynamical system acts on the $L$–torus for example, $M = [0,1]^L$, one can take the Cartesian product of the cosine noise (37) setting

$$
\mathcal{P}_N(\xi_1, \ldots, \xi_L) = C_N^L \cos^N(\pi \xi_1) \cos^N(\pi \xi_2) \cdots \cos^N(\pi \xi_L),
$$

where $\xi_k = x_k - y_k$ and $k = 1, \ldots, L$. This form of the additive noise was used in [15] to analyze a 2-dimensional system (a variant of the baker map), and to compare the spectral properties of the FP operator associated with the classical stochastic system with properties of the propagator of the corresponding quantum evolution. In such a case the deterministic limit of the classical noisy system, $K \to \infty$ is related to the classical limit, $\hbar \to 0$, of the corresponding quantum dynamics.

Note that for basis–Markov stochastic systems, the transition matrices $T$ exactly describe the action of the dynamical system with additive noise on densities. Thus our construction differs from an approach applied in [17–19], were a finite dimensional description of the density dynamics of a deterministic system was achieved by truncation of an infinite transition operator $P_f$ to the finite dimension $K$. The effect of such a truncation may also be regarded as a kind of noise depending on the matrix size $K$ and the base, in which $P_f$ is represented. On the other hand, in our case a suitable choice of the noise profile added to the deterministic system distinguishes a relevant basis, in which the FP operator of the perturbed system is finite.


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IX. APPENDIX: ISOSPECTRAL MATRICES

In this appendix we show that the matrix $D$ defined by Eq. (27) and used in [12, 13, 15] to represent the Frobenius-Perron operator and the stochastic transition matrix $T$ share the same non-zero part of the spectrum. We make use of a following algebraic result,

**Lemma.** Let $A$ be a square matrix of size $N \times N$ and $\vec{s}$ a vector of length $N$ containing only non-zero entries. Then the matrix

$$B_{jk} \equiv A_{jk} \frac{s_j}{s_k},$$

(54)

has the same spectrum as $A$.

(there is no summation over repeating indices).

**Proof:** To study equation $\det(B - \lambda \mathbb{1}) = 0$ we start analyzing an exemplary term $P^B$ of the determinant. It consists of a product of $N$ elements $B_{i,\sigma(j)}$, where $\sigma(i)$ stands for a certain permutation of the indices. The product of $N$ factors of the type $s_i/s_{\sigma(i)}$ is equal to unity, so that

$$P^B_{\sigma} = \prod_i B_{i,\sigma(i)} = \prod_i B_{i,\sigma(i)} \frac{s_1s_2\cdots s_N}{s_1s_2\cdots s_N} = \prod_i A_{i,\sigma(i)}. \quad (55)$$

Thus every term contributing to the free coefficient of the characteristic equation will be the same, $P^B_{\sigma} = P^A_{\sigma}$, hence these coefficients for both matrices $A$ and $B$ are equal. Since the diagonal elements of both matrices coincide, $B_{jj} = A_{jj}$, all terms forming the coefficients standing by an arbitrary power of $\lambda$ are the same for both matrices. Therefore characteristic equations for both matrices are equal and so are their spectra. □

Treating all non-zero elements of the vector $\tau_k, k = 1, \ldots, K$ as vector $\vec{s}$ we may apply the lemma to equation (36) and obtain equivalence of the spectrum of $T$ and the non-zero part of the spectrum of $D$. Since integrals (25) vanish for odd values of $m$, every second
column of $D$ is equal to zero, and the remaining $N/2$ eigenvalues of $D$ are equal zero.


