Complex flows mix efficiently, and this process can be understood by considering the stretching and folding of material volumes. Although many metrics have been devised to characterize stretching, fewer are able to capture folding in a quantitative way in spatiotemporally variable flows. Here, we extend our previous methods based on the finite-time curving of fluid-element trajectories to nonzero scales, and show that this finite-scale finite-time curvature contains information about both stretching and folding. We compare this metric to the more commonly used finite-time Lyapunov exponent, and illustrate our methods using experimental flow-field data from a quasi-two-dimensional laboratory flow. Our new analysis tools add to the growing set of Lagrangian methods for characterizing mixing in complex, aperiodic fluid flows.
One of the hallmarks of chaotic or turbulent fluid flows is rapid mixing. Vigorously stirred fluids mix so well because they both move material quickly and yet are constrained to remain in a finite amount of space. The chaotic mixing process can thus be conceptualized as the result of stretching—that is, the rapid separation of nearby packets of fluid—and folding—that is, the bending of packets of fluid to maintain the overall volume. Because these two processes are fundamental for mixing, they have been the subject of intense study; however, many more methods have been developed to characterize stretching than folding. Here, we describe a method based on the propensity of the trajectories of fluid elements to curve that allows us to quantify both stretching and folding at the same time. We compare this technique with a more common tool that isolates only stretching, and demonstrate our methods on data from an experimental flow.

1. INTRODUCTION

It is almost definitional to say that chaotic and turbulent flows mix efficiently: by a repeated process of stretching and folding material volumes\(^1\), chaotic advection\(^2\) leads to mixing that is significantly enhanced relative to diffusion alone\(^3\). Historically, this rapid mixing has been quantified and modeled using statistical metrics such as the relative dispersion of fluid elements\(^4\) or effective diffusivities\(^5\). These kinds of tools capture the global effects of the fluid advection on mixing and can provide a simple, high-level parameterization of its effects that is useful for modeling.

But even though transport and mixing are sped up by chaotic advection on average, this enhancement is rarely uniform; rather, it mirrors the complex spatiotemporal heterogeneity of the flow itself\(^6\). Even in a flow that is in general strongly mixing, there may be regions that are separated from the rest of the flow by dynamical barriers, and which mix only weakly. In the terminology of dynamical-systems theory, these weakly mixing regions may be described as being elliptic, while those parts of the flow that mix strongly and transport material rapidly are hyperbolic. Since chaotic and turbulent flows are typically highly unsteady and aperiodic, however, the instantaneous elliptic or hyperbolic character of a particular region of the flow may change rapidly in time. Thus, any useful partitioning of the flow field into strongly and weakly mixing regions must identify not only places that are elliptic or hyperbolic but those that maintain their character over some macroscopic time window. Such regions may be said to be coherent\(^7\), and are
often referred to as coherent structures.

Analyzing flows from the standpoint of coherent structures may allow us to move past global descriptions of transport and mixing, such as overall effective diffusivities or purely statistical models, to local characterizations. Such local information is often a very important complement to its global counterpart; in an oil spill in the ocean, for example, one is more interested in knowing which areas of the coastline will be affected than in the net spreading rate of the oil. Due to this goal, then, a vast set of methods has been devised to detect and characterize coherent structures in general unsteady flows. We focus here on methods that use primarily Lagrangian information; that is, techniques that are based on the information contained in the trajectories of fluid elements, typically computed over some finite time that specifies the window in which we desire coherence. Broadly, one can classify these methods into those that attempt to find the boundaries that separate coherent structures from the rest of the flow and those that try to find the structures themselves by set oriented methods.

We recently developed a Lagrangian diagnostic that we term the finite-time curvature (FTC) and that was designed to study shape-coherent sets (that is, those that retain their shape as they are advected). Here, we study the FTC field in more detail, and show that it contains a significant further information beyond shape coherence that can be used to quantify mixing. We compare the FTC to the more commonly known finite-time Lyapunov exponents (FTLEs) often used to identify Lagrangian coherent structures (LCSs), and show that the two contain similar and complementary, though not identical, information: FTLE can be thought of as the local propensity for stretching in a dynamical system, while FTC (appropriately extended to account for finite scale) additionally captures the local propensity for folding. Although stretching and folding often take place nearby each other in similar places, their spatial distribution is not identical; and, of the two, folding has been significantly less well studied. But since it is axiomatic that both stretching and folding are necessary for chaos, both kinds of information are needed to characterize chaotic mixing fully, and it is natural to ask about folding as well as stretching. Thus, the finite-scale FTC we introduce here usefully extends the toolbox that can be used to study mixing in complex flows from the Lagrangian standpoint.

We begin below by reviewing the FTLE in Section 2. In Section 3, we review the definition of the FTC, and generalize our previous methods to account for the effects of finite scale and resolution. We then illustrate our methods using experimental data from a quasi-two-dimensional laboratory flow in Section 4. Finally, we summarize our results in Section 5.
2. REVIEW OF FINITE TIME LYAPUNOV EXPONENTS

The finite-time Lyapunov exponent (FTLE) is a measure of local stretching in the flow. FTLEs are commonly used as indicators for hyperbolic LCSs, since in hyperbolic regions of the flow, stretching is locally maximal. Even though FTLEs have shortcomings for the detection of LCSs (primarily that they also detect regions of high shear), they remain perhaps the mostly widely used tool for studying Lagrangian coherence, particularly in experiments where more detailed information may not be available. Here, we briefly review the definition and properties of FTLEs, so that we may later compare and contrast them with FTC.

Given a velocity field \( u(x, t) \) on a manifold \( M \subset \mathbb{R}^d \), the trajectories \( x(t) \) of fluid elements are solutions of
\[
\frac{dx}{dt} = u(x, t),
\]
(2.1)
where \( x \in M \) and \( u(x, t) \) is at least \( C^2(M) \). Integrating this equation yields the flow map \( \phi^{t+\tau}_t : x(t) \mapsto x(t + \tau) \). The finite-time Cauchy–Green strain tensor of the velocity field along the trajectory \( x(t) \) is given by the symmetric, time-dependent, \( d \times d \) matrix
\[
J_\tau = \left[ \frac{d\phi^{t+\tau}_t x(t)}{dx} \right]^* \frac{d\phi^{t+\tau}_t x(t)}{dx},
\]
(2.2)
where \( A^* \) denotes the adjoint of \( A \). In the following, we assume that \( d = 2 \), but this assumption is not mathematically necessary.

If over a finite time interval \( [t, t + \tau] \), the minimum and maximum eigenvalues \( \lambda_{\text{min}}(\tau) \) and \( \lambda_{\text{max}}(\tau) \) of \( J_\tau \) satisfy the condition
\[
\ln \lambda_{\text{min}}(\tau) < 0 < \ln \lambda_{\text{max}}(\tau),
\]
(2.3)
then the canonical local material advective behavior is described by the evolution of circles into ellipses, where the major axis of the ellipse lies along the direction of instability. This condition implies that there is compression in one direction and expansion in the other along the trajectory. Such a trajectory in a time-dependent velocity field is referred to as a hyperbolic trajectory.

Recall that the spectral norm of the Jacobian \( \frac{d\phi^{t+\tau}_t x(t)}{dx} \) is given by
\[
\left\| \frac{d\phi^{t+\tau}_t x(t)}{dx} \right\|^2 = \lambda_{\text{max}}(\tau).
\]
(2.4)
Then the FTLE, which represents the maximum stretching at the point \( x(t) \) along the trajectory over a time \( \tau \), is given by

\[
\sigma_{\tau}(x(t)) = \frac{1}{|\tau|} \ln \left\| \frac{d\phi_{t+\tau}^t x(t)}{dx} \right\| = \frac{1}{|\tau|} \ln \sqrt{\lambda_{\max}(\tau)}. \tag{2.5}
\]

It is common to define the repelling and attracting LCSs to be the ridges of the FTLE that are local maxima in space when computed in forward time (\( \tau > 0 \)) and backward time (\( \tau < 0 \)), respectively\(^{29,30}\). Stated in terms relevant to our discussion here, the intensity of the FTLE field in a flow describes the propensity of ensembles of fluid elements that are nearby at time \( t \) to separate as the flow evolves.

Recall that, generally, a matrix “times” a circle yields an ellipse\(^{27,31}\). That is, from a base point, the “tips” of the set of all unit vectors in the plane form a circle, and the matrix times the set of each of these vectors forms an ellipse as the tips of these. See Fig. 1. Then the positive semi-definite matrix \( J_\tau \) in Eq. (2.5) as the strain formed from the variation of the flow, \( \frac{d\phi_{t} x(t)}{dx} \), along the orbit, \( x(t) \), can be shown\(^{31}\) to have eigenvectors pointing along the major axis of the ellipse, and correspondingly the singular values are the lengths of the major axis corresponding to the growth rate of errors. We will derive further interpretation of this stretching picture of infinitesimal action along orbits below, as related to infinitesimal folding.

3. REVIEW AND COMPUTATION OF FINITE TIME CURVATURE

In our recent work\(^{23}\), we developed a direct Lagrangian measure of “curvature propensity,” which we termed the maximum finite-time curvature (maxFTC, or simply FTC), and which we used to infer the location of shape-coherent sets\(^{24}\). Here, we show that there is further structure to be found in the same computed field.

The idea of a shape-coherent set is that “coherence” can be interpreted independently of the more commonly discussed direct notions of transport. That is, a set that maintains its shape to a high degree along its orbit can in a visceral way be considered to be coherent (specifically shape-coherent, in our terminology). With this notion in hand, simple geometric reasoning allows one to observe that the way to maintain shape is to maintain the curvature of the boundary of the set. Thus, investigating regions of space that have a very low propensity to develop curvature may reveal such sets.

We therefore defined the maximum finite-time curvature (maxFTC) \( K_{t_0}^{t_0+\tau}(z) \) for a point \( z \) in
a plane $M \subset \mathbb{R}^2$ under a flow $\phi_{t_0}^{t_0+\tau}$ over the time interval $[t_0, t_0 + \tau]$ to be

$$K_{t_0}^{t_0+\tau}(z) = \lim_{\varepsilon \rightarrow 0} \sup_{\|v\|=1} \kappa(\phi_{t_0}^{t_0+\tau}(l_{\varepsilon,v}(z))),$$

(3.1)

where

$$l_{\varepsilon,v}(z) := \{ \dot{z} = z + \varepsilon s v, |s| < 1 \},$$

(3.2)

and $v$ is a unit vector. So, $l_{\varepsilon,v}(z)$ is a small line segment passing through the point $z = (x, y)$, when $\varepsilon \ll 1$.

FIG. 1: On the finest scale, a small circle of radius $\varepsilon$ along the orbit of a point $x(t)$ evolves to an ellipse, and this estimate describes that the linearity dominates, in agreement with the commonly understood concept that “matrix times circle yields ellipse.” However the geometry of the intermediate scale where this estimate differs from observation is described in Fig. 3, and the suggestion of Fig. 4 is that this transition between fine and intermediate scale may occur at smaller length scales than suggested by the common practice of studying only the linear terms.

In practice it is a computational challenge as to how we implement both the optimization step and the limit step ($\varepsilon \rightarrow 0$) in Eq. (3.1). As we discuss further below, on the finest spatial scales there is a strong correlation between FTC and FTLE; on intermediate scales, however, they differ, as the FTC can uncover the nonlinearities in the flow. To distinguish these “finest” scales and “intermediate” scales precisely, we choose an $\varepsilon > 0$ and define a Finite-Scale Finite-Time Curvature.
(fsFTC) as

$$K_{t_0}^{t_0+t}(z, \varepsilon) = \sup_{\|v\|=1} \kappa(\phi_{t_0}^{t_0+t}(l_{\varepsilon,v}(z))). \quad (3.3)$$

Contrasting to the FTC in Eq. (3.1), notice that the limit is omitted, and this argument explicitly includes the independent variable $\varepsilon > 0$. Furthermore and generally, for smooth flows the sup should be realized, and so can be replaced by a max operation. Note that the image of the line segment $l_{\varepsilon,v}(z)$ is generally a curve $\phi_{t_0}^{t_0+t}(l_{\varepsilon,v}(z))$, and the role of the limit is to isolate the curvature to the locality of the image of the center point, at $\phi_{t_0}^{t_0+t}(z)$. In practice, the finite scale implicit in where and how the curvature is estimated in the neighborhood of $\phi_{t_0}^{t_0+t}(z)$ makes a difference for what is observed.

A. FTC on Intermediate Scales

The most straightforward interpretation of the estimation of the FTC by the fsFTC formula in Eq. (3.3) is by what we may refer to as the brute-force computation. First choose $n$ sample vectors $v$ pointed around a unit circle and uniformly spaced, and scale these by a small but finite distance $\varepsilon > 0$ to form $\varepsilon v$. The direction of each unit vector $v$ is specified by an angle $\gamma$. Then form $n$ triplets of points

$$A = z - \varepsilon v, B = z, C = z + \varepsilon v, \quad (3.4)$$

approximating the line-segment instance of Eq. (3.2). Each can then be mapped forward under the flow to new locations

$$A' = \phi_{t_0}^{t_0+t}(z - \varepsilon v), B' = \phi_{t_0}^{t_0+t}(z), C' = \phi_{t_0}^{t_0+t}(z + \varepsilon v) \quad (3.5)$$

after a finite time $\tau$; see Fig. 3 for an illustration. It is a fact of geometry that there is a unique circle passing through any three (non-colinear) points $A', B', C'$ called the Menger circle. Calling the radius of this circle $R_{A',B',C'}$, the Menger curvature $^{32-34}$ is defined as $\kappa_M(A', B', C') = 1/R_{A',B',C'}$. The Menger curvature may be computed by the convenient classical formula,

$$\kappa_M(A', B', C') = \frac{c}{2 \sin \gamma}. \quad (3.6)$$

See Fig. 2. Here $\gamma$ may be chosen as any one of the 3 angles of the triangle $A'B'C'$ described by the three points and $c$ is the length of the corresponding opposite side. Note that order of labeling
is not important, as the circle is uniquely defined by the points.

The unit vectors \( \mathbf{v} = (\cos(s), \sin(s)) \) for each \( s \in [0, 2\pi) \) specify points \( (A, B, C) \) and their images \( (A', B', C') \) in Eqs. (3.4)-(3.5) for each angle. We may then write \( \kappa_M(s, \varepsilon) := \kappa_M(A', B', C') \).

Referring to Fig. 3 for labeling, each of the \( n \) uniformly sampled points around a circle of radius \( \varepsilon \), \( s_i = ih, \ h = 2\pi/n, \ i = 0, 1, \ldots (n - 1) \), yields a sample of the function by points \( (s_i, \kappa_M(s_i, \varepsilon)) \).

See Fig. 3. From this finite sample we may estimate

\[
K_{t_0 + \tau}^t(z, \varepsilon) \approx \max_i \{\kappa_M(s_i, \varepsilon)\}, \tag{3.7}
\]

for fixed \( \varepsilon \).

\[\text{FIG. 2: Menger curvature of three points } A, B, C \text{ is defined in terms of the radius of the unique circle passing through these points by formulas, Eqs. (3.4) and (3.6).}\]

Proceeding more carefully, but still on a finite intermediate scale with \( \varepsilon > 0 \), we can estimate the fsFTC itself, which then will give us an estimate of the true FTC. Unlike the above procedure, where we simply selected the maximal value from a large sampling, a more efficient and accurate way to optimize any function is to use a standardized optimization algorithm such as the gradient descent method\(^\text{35}\). This approach will work if the flow \( \phi_{t_0 + \tau}^t \) is sufficiently smooth at \( z \). It then follows that the fsFTC \( K_{t_0 + \tau}^t(z, \varepsilon) \) can be estimated by the optimal Menger curvature \( z \) for a given small \( \varepsilon \), so that

\[
K_{t_0 + \tau}^t(z, \varepsilon) \approx \max_{s \in [0, 2\pi]} \kappa_M(s, \varepsilon)), \tag{3.8}
\]
FIG. 3: Curvature due to sampling 3 opposing points along a line for \( n = 33 \) points around a small circle, according to Eqs. (3.4)-(3.5), and estimating curvature of the image by the Menger curvature by Eq. (3.6). The Intermediate scale gives rise to (a) curvature as a function of samples \( s \) around the circle that differs significantly from the fine scale prediction suggested by curvatures of the ellipse, as seen by the limit curve in Fig. 4 extrema that occur at the major and minor axis of the ellipse. (b) In this intermediate scale, the image of a circle is not simply an ellipse as suggested by Fig. 1, but rather may have a significant folding nonlinearities shown here. In the regions of phase space where the folding is prominent, this effect can be significant.

We already noted that the image of the line segment \( l_{\varepsilon,v}(z) \) is the curve \( \phi_{l_0}^{t_0+\tau}(l_{\varepsilon,v}(z)) \); the Menger curvature \( \kappa_M(A', B', C') \) is used here as an estimator of the curvatures in the neighborhood of \( \phi_{l_0}^{t_0+\tau}(l_{\varepsilon,v}(z)) \), assuming small \( \varepsilon \) and smooth \( \phi_{l_0}^{t_0+\tau} \). See Fig. 3. The function \( \kappa_M(s, \varepsilon) \) shown in Fig. 3a shows complex structure that is due to the finite, nonzero \( \varepsilon > 0 \). In contrast, as shown below, for the finest scales where \( \varepsilon \ll 1 \), \( \kappa_M(s, \varepsilon) \) displays only simple periodic behavior of the
ellipse with respect to \( s \). We emphasize that this observation is not simply an issue with estimation for nonzero \( \varepsilon \); as we demonstrate below, the transition between intermediate and fine scales may occur at extremely small \( \varepsilon \) compared with observable scales in experiments. More interestingly, however, we also show that we can use the inherent finite-scale nature of the fsFTC to infer information about the flow nonlinearities.

B. FTC on the Finest Scales

For fine scales with \( \varepsilon \ll 1 \), small enough that the image of a circle of radius \( \varepsilon \) about \( z \) is essentially an ellipse with minimal non-affine distortion, the computation of FTC simplifies to an analytic expression we describe here. In this case, the smallness of the scale effectively linearizes the action of the flow, meaning that the FTC can no longer capture the propensity of the flow to fold curves. However, as we discuss below, it is often the intermediate-scaled folding that presents the interesting features.

In Fig. 6, we illustrate a unit circle (representing the \( \varepsilon \)-radius circle with its center at \( z \)), a concentric circle of radius \( r > 1 \), and their images. Applying the variational derivative of Eq. (2.1) along orbits yields the linearization of the flow as the matrix \( \frac{d\phi(t)}{dx} \), and the corresponding positive-semidefinite form, the strain \( J_\tau \) described in Eqs. (2.1)-(2.2). For simplicity, we denote the Jacobian matrix as \( D = \frac{d\phi(t)}{dx} \) and the strain as \( J_\tau = D^*D \). We can apply a singular value decomposition to \( D \), obtaining \( D = U\Sigma V^* \). \( \Sigma = diag(\sigma_1, \sigma_2) \) is the diagonal matrix whose nonzero entries are the major and minor axis lengths of the image ellipse of the unit circle shown; equivalently, they are the eigenvalues of \( J_\tau \). \( V = [v_1|v_2] \) is the orthogonal matrix whose orthogonal column vectors are oriented along the major and minor axes of the image ellipse, and are also the right eigenvectors of \( J_\tau \), which, without loss of generality, we have illustrated for convenience to be oriented along the \( x \) and \( y \) axis and we take \( z = 0 \). We have included a point \( w = (\sigma_1, 0) \), without loss of generality at angle \( s = 0 \), along the major axis \( v_1 \). \( w = (r\sigma_1, 0) \) is the corresponding point on the concentric ellipse shown. We also illustrate in Fig. 6 a red line segment of length \( 2\delta \) that gives the Menger image with sides \( \delta \) through \( w \).

Let us now consider the development of curvature by the local dynamics as suggested by Fig. 6. The Menger curvature through the points on the ellipse shown in Fig. 6, \( A' = (\sigma_1 r \cos(s), \sigma_2 r \sin(s)) \), \( B' = w = (r\sigma_1, 0) \), and \( C' = (\sigma_1 r \cos(s), -\sigma_2 r \sin(s)) \), estimates the curvature of the ellipse at \( w \).
Specifically,
\[ \lim_{s \to 0} \kappa_M(A', \bar{w}, C') = \kappa(\bar{w}) = \frac{\sigma_1}{r\sigma_2^2}. \]  
(3.9)

The first part of this equality follows from the continuity property of Menger curvature: for smooth curves, if the three points \(A', \bar{w}, C' \to \bar{w}\), then the Menger curvature limits to the curvature at the point. The computation of the second part of the equality is detailed in Appendix 6. In stating the next limit, notice that Eq. (3.10) differs from Eq. (3.9) in that \(\bar{w}\) and \(w\) are similarly positioned, but on concentric ellipses. Considering concentric circles of radius \(r > 1 > 0\), the Menger curvature of the points shown in Fig. 6, \(A' = (\sigma_1 r \cos(s), \sigma_2 r \sin(s))\), \(B' = w = (\sigma_1, 0)\), and \(C' = (\sigma_1 r \cos(s), -\sigma_2 r \sin(s))\), are estimated by the curvature of the ellipse at \(\bar{w}\), since \(\bar{w} \to w\) as \(r \to 1\), so that
\[ \lim_{\bar{w} \to w} \kappa_M(A', \bar{w}, C') = \kappa(w) = \frac{\sigma_1}{\sigma_2^2}. \]  
(3.10)

It then follows that if the flow \(\phi_{t_0}^{t+\tau}\) at \(z\) is continuously differentiable, the FTC is estimated by the limit of curvatures as
\[ \text{K}_{t_0}^{t+\tau}(z) = \lim_{\varepsilon \to 0} \text{K}_{t_0}^{t+\tau}(z, \varepsilon) \sim \max_{s \in [0, 2\pi]} \kappa_M(s, \varepsilon) \sim \max \left\{ \frac{\sigma_1}{\sigma_2^2}, \frac{\sigma_2}{\sigma_1^2} \right\}. \]  
(3.11)

where the terms are, in order, the FTC, the fsFTC, the maximum Menger curvature evolved from center of the circle according to Eq. (3.8), and the maximum ratio of the singular values. If furthermore the flow is area-preserving, then \(\sigma_1 = 1/\sigma_2\). Hence, in this special case, \(\frac{\sigma_1}{\sigma_2^2} = \sigma_1^3\), \(\frac{\sigma_2}{\sigma_1^2} = \sigma_2^3\), and so \(K_{t_0}^{t+\tau}(z) = \sigma_2^3\). The arguments supporting these statements in Appendix 6 are straightforward. In the next section we interpret consequences both for the relationship of the study of curvature evolution to coherence, hyperbolicity, and the contrast of scales.

C. Contrast of Scales

As noted above, the FTC should be interpreted in different ways depending on the spatial scale on which it is applied. Specifically, consider Fig. 4, where we plot profiles of the curvatures around the circle for decreasing \(\varepsilon = 10^{-1}, 10^{-2}, ..., 10^{-7}\). We indicate the transition that separates the intermediate scale from the fine scale at roughly \(\varepsilon = 10^{-2}\), where the local circle of line segments exhibits significant folding, from the finest scales, where it shows the simpler curvature of ellipses behavior predicted by Eqs. (3.9)-(3.11),(6.2). Figure 4 illustrates the two somewhat different types of information contained in the FTC. To restate these two aspects more descriptively, minimal
values of the FTC mark regions in the flow where materials curves change spatiotemporally slowly. This feature of the FTC was the original reason we developed the analysis\textsuperscript{23}, as it can be used to locate shape-coherent sets\textsuperscript{24}. The converse, however, also holds: maximal values of the FTC indicate regions where material curves most rapidly develop curvature, and thus the places in the flow where the shape of a set changes most rapidly.

How can we understand these opposite aspects in more familiar terms? The description in terms of shape coherence suggests that it is the elliptic-like behavior of the flow that causes the low troughs of the FTC field. Hyperbolic material curves, however, will cause any transverse curve of material to deform rapidly, suggesting that ridges of the FTC field likely indicate hyperbolic behavior. Thus, the single FTC computation can be used to locate both hyperbolic (ridges) and elliptic (valleys) regions of the flow field.

The analysis above in Eqs. (3.9)-(3.11) affirms that there is a strong relationship between FTC and FTLE on fine scales, since both are related to the singular values of the Jacobian matrices along the flow. But, on the finest spatial scale, Eq. (3.11) shows that FTC is related to the cube of the FTLE. This feature has an important consequence: one would expect that ridges of the FTC field will be more prominent and will appear for shorter advection times than ridges of the FTLE field. We expect, heuristically even if not in detail, that this feature will be retained even on coarser scales since the general notion that hyperbolicity emphasizes deformation of material curves is generally understood by consideration of a transverse curve that samples the flow becomes
exponentially deformed. The more rapid appearance of structure in the FTC field makes it a promising candidate for forecasting applications, since less future-time information is needed to locate hyperbolic regions.

4. EXPERIMENTAL DEMONSTRATION

To illustrate the similarities and differences between the FTC and the FTLE in a practical example, we measured both using experimental data from a quasi-two-dimensional laboratory flow. The details of this experiment have been described previously.\textsuperscript{36–38} Briefly, we used a thin electromagnetically driven layer of an electrolytic fluid to produce nearly two-dimensional flow. The working fluid was a layer of salt water (16\% NaCl by mass) measuring $86 \times 86 \times 0.5 \text{ cm}^3$ that rested on a glass plate coated with a hydrophobic wax. A square array of permanent neodymium-iron-boron magnets with vertical dipole moments and a lateral spacing of $L_m = 2.54 \text{ cm}$ lies beneath the glass; here, the magnets were arranged in stripes of alternating polarity. By running an electric current (of 1.25 A for the experiments analyzed here) through the salt water, we generated a Lorentz body force on the fluid that set it into motion. The Reynolds number $\text{Re} = U L_m / \nu$, based on the in-plane root-mean-square velocity $U$, the magnet spacing, and the kinematic viscosity $\nu$ was 270.

We measured the velocity field using particle tracking velocimetry (PTV). The electrolyte was seeded with 50-\textmu m-diameter fluorescent polystyrene microspheres; as the spheres are somewhat less dense than the electrolyte, they rise to its surface. To eliminate long-range surface-tension-driven forces between the floating particles, we floated a layer of pure water 5 mm deep on top of the electrolyte. We imaged the motion of the particles in the central $31.7 \times 23.6 \text{ cm}^2$ (roughly $12.5 \times 9 L_m$) of the apparatus using a 4 megapixel IDT MotionPro M5 camera at a rate of 60 frames per second. We then tracked about 30 000 particles per frame using a multi-frame predictive tracking algorithm,\textsuperscript{39} and computed time-resolved velocities from the trajectories by convolving them with a smoothing and differentiating kernel.\textsuperscript{40} Finally, we used the instantaneous velocities of all the tracked particles to create velocity fields by projecting the data onto the eigenmodes of a streamfunction,\textsuperscript{36} removing noise from the data and ensuring that the measured velocity fields are reliably two-dimensional.

In Fig. 5, we show both the FTC and the FTLE computed over the same time window for the experiment. Even though it is high, the experimental resolution is still finite; thus, the FTC plotted here should be interpreted as the fsFTC, and we may expect it to display finite-scale
FIG. 5: A a) FTLE field and a b) FTC field of the same dataset derived from a unsteady flow from a quasi-two-dimensional laboratory flow. Note that there are both similarities in the ridges as both suggestive of hyperbolicity our discussion, and differences, particularly regarding the dark blue troughs seen in the FTC field suggesting rigidity. Integration time for both is, $5s \approx 3T_L$, where $T_L = L_m/U$ is the eddy turnover time.
features—most importantly, the folding propensity of the flow.

The FTLE (Fig. 5(a)) contains the features that are typically seen in this kind of flow. Most of the field is filled with relatively small, though still positive, values, indicating that the full field mixes chaotically, but relatively weakly. The FTLE also reveals very strongly stretching regions that are nearly co-dimension one; these line-like regions should approximately correspond to the Lagrangian coherent structures in the flow. In the cores of the areas in between the FTLE ridges, the FTLE values go negative, marking regions that are elliptic in character.

Comparing the FTLE with the fsFTC shown in Fig. 5(b), it is clear that the fsFTC reveals some of the same hyperbolic features. And indeed, for the same integration time (which was fixed for both panels in Fig. 5), these features are sharper for the fsFTC: the colormap for the fsFTC in Fig. 5(b) is on a logarithmic scale, while it is on a linear scale for the FTLE in Fig. 5(a). This kind of rapid convergence is very useful in experimental or observational applications, where long records of future flow-field information may not be available. But more than this simple difference, the fsFTC reveals additional structure. Regions where the fsFTC is small are not simply not stretching; they are also, by construction, not folding, and are thus evolving as semi-rigid bodies over the time window used to calculate the fsFTC. This kind of behavior should be contrasted with structures such as the Great Red Spot on Jupiter, where a macroscopic region of the flow is elliptic in that it is separated from the rest of the flow by a persistent transport barrier, but where the flow inside the barrier is still highly turbulent. For macroscopic areas where the fsFTC is small throughout, we expect that the internal flow is not only distinguished from the external flow, but is also evolving simply and not in a turbulent or chaotic fashion. Of additional note are the thin lines of low fsFTC values; these minimal curves of the fsFTC behave as they are nearly rigid, and so they can rotate as time evolves but cannot bend.

5. SUMMARY

Chaotic mixing can be schematically as a repeated process of the stretching and folding of material volumes; both of these processes are required to produce efficient mixing in a volume-preserving flow. By explicitly considering the effects of finite resolution on the finite-time curvature, we have shown here that this fsFTC naturally contains information about both stretching and folding. Thus, we have demonstrated that the FTC can be used to indicate more than the shape coherence it was designed for. We illustrated the utility of the fsFTC by computing both it and the more common FTLE on experimental data from a laboratory flow, showing that the fsFTC
is simply computable and that it tends to reveal sharp features more rapidly than the FTLE. Thus, the fsFTC we have introduced is a valuable addition to the growing collection of Lagrangian methods that can be used to explore and characterize mixing and transport in complex flows.

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6. APPENDIX

Here we prove the statement in Eqs. (3.9)-(3.10) regarding the ratio of singular values. It is well known that the image of a circle (that is, the set of all unit vectors) is an ellipse under the action of a general 2 by 2 matrix $D$. Curvature at a point is defined as the inverse of the radius of the osculating (tangent) circle to a curve at the point. Thus, a unit circle has curvature $\kappa = 1$ by definition. Let the image of the circles shown in Fig. 6 be the ellipses shown, where without loss of generality the body axes are aligned with the coordinate axes for convenience as shown, and the major axes $v_1, v_2$ have lengths $\sigma_1, \sigma_2$ from the SVD $D = U\Sigma V^*$. The inner ellipse shown can be
written implicitly as \( \frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} = 1 \), or parametrically as

\[
\gamma(s) = <\sigma_1 \cos s, \sigma_2 \sin s >.
\]  

(6.1)

By a standard computation of the curvature in terms of a parametrically represented curve, it follows that

\[
\kappa(s) = \frac{|\gamma'(s) \times \gamma''(s)|}{|\gamma'(s)|} = \frac{\sigma_1 \sigma_2}{|\sigma_1^2 \sin^2(s) + \sigma_2^2 \cos^2(s)|^{3/2}}.
\]  

(6.2)

Solving \( \kappa'(\hat{s}) = 0 \) for the critical points \( \hat{s} \) yields \( \hat{s} = 0, \frac{\pi}{2} \), and hence the extrema of curvature are found by substitution into Eq. (6.2) to be

\[
\kappa(\hat{s}) = \{\frac{\sigma_1}{\sigma_2^2}, \frac{\sigma_2}{\sigma_1^2}\}.
\]  

(6.3)

Likewise, the curvatures of the inner and outer ellipses at \( \mathbf{w} \) and \( \overline{\mathbf{w}} \), respectively, are \( \kappa(\mathbf{w}) = \frac{\sigma_1}{\sigma_2^2} \) and \( \kappa(\overline{\mathbf{w}}) = \frac{\sigma_2}{\sigma_1^2} \). The limit in Eq. (3.9) follows immediately since the cosine and sine functions in the parametric equation of an ellipse (Eq. (6.1)) are continuous. The limit in Eq. (3.10) follows by noting the continuity of Menger curvature and that \( A', C' \to \overline{\mathbf{w}} \) as \( r \to 0 \) and \( \mathbf{w} \to \overline{\mathbf{w}} \).

The argument of Eq. (3.11) follows by realizing that a spatially continuously differentiable flow \( \phi_{t_0+\tau}^{l_0} \) has Jacobian matrices that change continuously with respect to space. Therefore, in reference to Fig. 6, the manner in which a line segment folds maximally according to the curvature of an ellipse at a point on the major axis, by the above Eq. (6.3), applies also to the center of the ellipse and a line segment through it since this is on the major axis of another nearby point with a nearby Jacobian matrix.

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