(Chapter head:)Sequences and Sequential Limits

1 Sequences

The basic idea of a sequence of numbers is an unending list of numbers in a specific order. The individual numbers are called the *terms* of the sequence. The sequence consists of the first term, the second term, the third term and so on indefinitely. More formally:

Definition 1 *A sequence is a function whose domain is the set of positive integers.*

Sometimes it is convenient to consider sequences defined on the *nonnegative* integers. Such a sequence starts with its "zero-th" term.

Notation Typically a sequence is denoted by (a_n) where the individual terms are denoted

$$a_1, a_2, a_3, \ldots$$

The *general* or n-th term is denoted by a_n. This subscript notation can be regarded as a shorthand for $a(n)$ where a is the function defining the particular sequence under consideration.

Examples Simple examples of sequences, and their first few terms and general term are:

(1)	$a_n = 1$	1	1	1	1
(2)	$b_n = n$	1	2	3	n
(3)	$x_n = (-1)^n$	-1	1	-1	$\begin{cases} -1 \text{ if } n \text{ is odd} \\ +1 \text{ if } n \text{ is even} \end{cases}$
(4)	$u_m = \dfrac{1}{m^2}$	1	$\frac{1}{4}$	$\frac{1}{9}$	$\dfrac{1}{m^2}$

These are all examples of sequences which are defined by a simple algebraic formula. However there are many situations in which sequences are more naturally defined *recursively* which is to say that each term is obtained according to some rule *from its predecessors.*

Often sequences can be described either way equally easily. For example the first three sequences above could all be defined recursively as follows:

(1)	$a_n = a_{n-1};$	with	$a_1 = 1$	gives	$a_n = 1$
(2)	$b_n = 1 + b_{n-1};$		$b_1 = 1$		$b_n = n$
(3)	$x_n = -x_{n-1};$		$x_1 = -1$		$x_n = (-1)^n$

Note Both the recursive equations and the *initial conditions* are essential to these definitions. Changing the initial value x_1 to -2 for example would define the sequence $2, -2, 2, -2, \ldots$ rather than $-1, 1, -1, 1, \ldots$

The last of the earlier examples is not easy to define recursively however. Similarly there are sequences which can be defined recursively but are not easily converted to a simple algebraic formula for the terms.

Examples Some sequences defined recursively:

1. The *Fibonacci sequence* is defined by

$$F_1 = F_2 = 1;\ F_{n+1} = F_n + F_{n-1} \quad (n > 1)$$

   It has its first few terms: $1, 1, 2, 3, 5, 8, 13, 21, 34$. Here each new term is the sum of its two predecessors. It is obtained by setting (in turn) $n = 2, 3, 4, \ldots$ in the *recurrence relation* $F_{n+1} = F_n + F_{n-1}$ and using the known values for previous terms.

2. The first few terms of the sequence given by

$$a_0 = 0;\ a_n = \sqrt{2 + a_{n-1}} \quad (n \geq 1)$$

   are $0, \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}$

The first of these can be "solved" to obtain a (complicated) algebraic formula for the terms. (However this would not be useful for any current purpose.) The second example does not even have such a "solution". Such sequences are nonetheless perfectly meaningful and we shall later investigate what happens to the terms of this sequence when n gets large – that is we shall discuss the "convergence" of such sequences.

Before looking at the formal question of convergence of sequences, we consider the computer generation of several terms of a sequence. This can often be a useful first step in analyzing the behavior of a particular sequence.

MATLAB Examples

1. Generate a table of the first 20 terms of the sequence $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}}$

```
EDU» for n=1:20
  a(n)=sqrt(n)/(1+sqrt(n));
end
EDU» [(1:20)',a']
ans =
  1 0.5
  2 0.58579
  3 0.63397
  4 0.66667
  5 0.69098
  6 0.7101
  7 0.72571
  8 0.7388
  9 0.75
 10 0.75975
 11 0.76834
 12 0.77599
 13 0.78287
 14 0.7891
 15 0.79479
 16 0.8
 17 0.80481
 18 0.80926
 19 0.81339
 20 0.81726
```

An alternative method would be to use MATLAB array arithmetic:

```
EDU» n=1:20;
EDU» a=sqrt(n)./(1+sqrt(n));
EDU» [n',a']
```

generates exactly the same output.

2. Generate a list of the first 10 terms of the recursive sequence $a_1 = -1/2$; $a_{n+1} = -a_n/2$ $(n = 1, 2, \ldots)$

```
EDU» a(1)=-1/2;
EDU» for n=1:9
 a(n+1)=-a(n)/2;
end
EDU» a
a =
 Columns 1 through 6
 -0.5 0.25 -0.125 0.0625 -0.03125 0.015625
 Columns 7 through 10
 -0.0078125 0.0039062 -0.0019531 0.00097656
```

(Here the actual MATLAB output is illustrated. Obviously the appearance of this output can be improved easily in other software or by the use of a *column* vector.)

3. Generate all the Fibonacci numbers smaller than 1000. (How many do you think there will be?)

Here the recurrence relation uses two previous members of the sequence. The solution below generates them as a MATLAB vector.

```
EDU» F(1)=1; F(2)=1; n=2;
EDU» while F(n)<1000
 n=n+1;
 F(n)=F(n-1)+F(n-2);
end
EDU» F
```

gives the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597 so that we have actually listed one more term than was required. To get an idea of how fast these terms are increasing we can use MATLAB's `plot` command

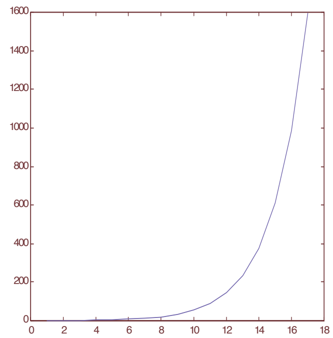
```
EDU» plot(F)
```

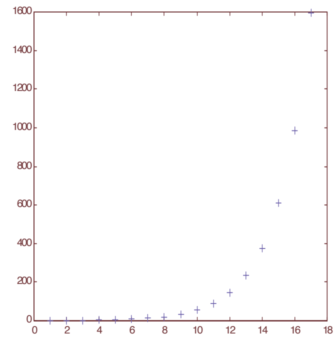
which generates the first plot below.

This gives the misleading impression that the sequence is a continuous function because MATLAB's default plot style connects the points. A better picture of this *sequence* is obtained by

```
EDU» plot(F,'+')
```

which results in the second "discrete" plot.





4. Often we are not interested in getting a complete list of terms of a sequence but in the question of whether or not the sequence converges. One way we might test this computationally is by generating terms until they "settle down". In this example we generate terms of the sequence given by $a_0 = 0$; $a_n = \sqrt{2 + a_{n-1}} \quad (n \geq 1)$ until two successive terms differ by less than 10^{-4}.

```
EDU» olda=0; newa=sqrt(2+olda);
EDU» while abs(newa-olda)>1e-4
 olda=newa;
 newa=sqrt(2+olda);
end
EDU» format long
EDU» [olda,newa]
ans =
 1.99996235056520 1.99999058761915
```

Note here that we do not try to generate a vector containing these terms. Just the most recent two terms, called `olda` and `newa` respectively are retained. We see that the last two terms generated indeed differ by less than the desired *tolerance* 10^{-4} which is easily written as `1e-4` in MATLAB.

1.0.1 Exercises

1. Give a recursive definition of each of the following sequences:

(a) $(2n-1)$ b. $\left(1/3^{n-1}\right)$ c. $(1/n!)$

d. $\left(\dfrac{(-1)^{n+1}}{n}\right)$ e. (2) f. $\left(\dfrac{(-2)^n}{(n+1)!}\right)$

2. Use MATLAB to generate tables of the first 25 terms of each of the sequences:

   a. $a_n = 1/n^2$ b. $b_m = -1/\sqrt{n}$ c. $c_n = n \sin(1/n)$

   d. $a_n = \dfrac{\sin n}{n}$ e. $x_r = \dfrac{r-1}{2r+1}$

   f. $y_n = \left(\dfrac{(-2)^n}{(n+1)!}\right)$ (Use the recursive definition found in Ex. 1(f).)

3. Modify the code in Example 3 above to print the ratios of successive Fibonacci numbers.

   (**Extra credit**: experiment with other initial conditions for the Fibonacci sequence and see what happens to these ratios.)

4. Use MATLAB programs to print the first 20 terms of each of the recursively-defined sequences below:

   (a) $a_1 = 1; \quad a_{n+1} = 1 + 1/a_n \quad (n = 1, 2, \ldots)$ (Compare your answers with those for Ex 3.)

   (b) $b_1 = 1; \quad b_n = \dfrac{2b_{n-1}+5}{b_{n-1}+2} \quad (n > 1)$

   (c) $c_1 = 5; \quad c_{n+1} = \sqrt{c_n} \quad (n \geq 1)$

   (d) $a_1 = 1, a_2 = 2; \quad a_{n+1} = a_n + 1/a_{n-1} \quad (n > 1)$

   (e) $a_1 = 1, b_1 = 2; \quad a_{n+1} = \sqrt{a_n b_n}, b_{n+1} = (a_n + b_n)/2 \quad (n \geq 1)$

5. For each of the sequences in Ex 4, what do you think happens as n gets very large? (Do the terms "settle down"? And if so, to what "limit"?)

2 Convergence of Sequences

Different sequences exhibit different types of behavior as n gets larger and larger, or " as $n \to \infty$". For example the terms of the sequence $(1/n)$ get ever closer to 0, while the terms of the Fibonacci sequence grow ever faster and "go off to infinity" and $(-1)^n$ oscillates between ± 1.

Sequences whose terms accumulate about some fixed number are said to "converge". We give a precise definition of this concept of convergence below. First, though, we relate it to our intuitive idea.

As an example, consider the sequence defined by

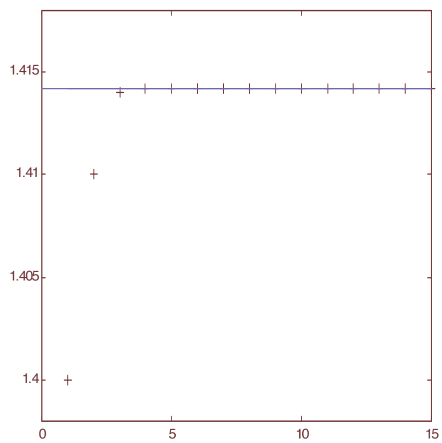
$$a_n \text{ is the } n\text{-decimal place expansion of } \sqrt{2}$$

That is $a_1 = 1.4, a_2 = 1.41, a_3 = 1.414, a_4 = 1.4142, a_5 = 1.41421, a_6 = 1.414214, a_7 = 1.4142136, a_8 = 1.41421356$ and so on. Fairly clearly the terms of this sequence are getting steadily closer to the "limit" $\sqrt{2}$. Moreover, they get *arbitrarily close* to this limit. That is, however small a "tolerance" we set, eventually all remaining terms of the sequence will be within that distance of $\sqrt{2}$.

For this particular sequence we can easily *bound* the error in using the n-th term as an approximation of the limit $\sqrt{2}$. If a_n is $\sqrt{2}$ correct to n decimal places, then the error satisfies

$$\left|a_n - \sqrt{2}\right| < 0.5 \times 10^{-n}$$

Graphically:



Here the crosses represent the terms of the sequence and the horizontal line is $y = \sqrt{2}$. The terms rapidly settle onto the line representing the limit.

This example illustrates all the important basic aspects of convergence:

- The terms get closer and closer to the limit
- The terms get (arbitrarily) close to the limit, and
- Eventually they stay arbitrarily close to the limit

To illustrate the differences between these, consider the sequence $\left(\dfrac{n}{n+1}\right)$, whose first few terms are $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$. It should be fairly clear that this sequence will eventually settle down to limit 1. Certainly we can see that the terms are getting closer to this limit – *but* they are also getting closer to 2, or any other number greater than 1 for that matter. This shows why we need the definition to say more than just that the terms get closer to the limit. We need them to get *arbitrarily close* to the limit. In this case the terms can be written as $1 - \dfrac{1}{n+1}$. Therefore after 10^N terms all subsequent terms will be within

10^{-N} of the limit. That is they get arbitrarily close – *and they stay there*. The need for the last condition, is that a convergent sequence may settle down near a fixed number for some time and then go off elsewhere – if it does, that fixed number is not its limit.

These conditions are formalized as

Definition 2 *A sequence (a_n) converges to the limit L if for every (small) $\varepsilon > 0$ there exists a number N such that*

$$|a_n - L| < \varepsilon \text{ for all } n > N$$

In this case we write

$$a_n \to L \text{ as } n \to \infty$$

or, equivalently

$$\lim_{n \to \infty} a_n = L$$

In this definition, ε is the tolerance we discussed above, the inequality $|a_n - L| < \varepsilon$ expresses the fact that the terms get arbitrarily close to the limit while the insistence that this holds *for all* $n > N$ is the insistence that the terms of the sequence eventually (once $n > N$) stay within the tolerance.

Note The condition $|a_n - L| < \varepsilon$ can often be usefully rewritten as $L - \varepsilon < a_n < L + \varepsilon$. We shall see examples of each being the convenient form to use for different sequences. Each simply says that the terms a_n are within the tolerance ε of the limit L.

As a first example, we shall prove that the above sequence where a_n is the n-decimal place expansion of $\sqrt{2}$ converges to the limit $\sqrt{2}$. (The point of proving this apparently obvious fact is that the algebra should not get in the way of understanding the proof.) The basic structure of the proof is that given an arbitrary small positive number $\varepsilon > 0$, we must find a "threshold" value N such that *all* the rest of the terms of the sequence for $n > N$ satisfy $|a_n - L| < \varepsilon$. So:

Let $\varepsilon > 0$. If we write down the decimal form of ε, we can count the number of 0's after the decimal point. Call this number N. It follows that $10^{-(N+1)} \le \varepsilon < 10^{-N}$. Now if $n > N$ then the error satisfies

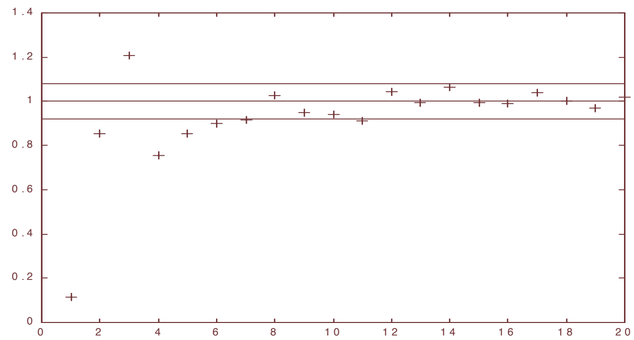
$$\left|a_n - \sqrt{2}\right| < 0.5 \times 10^{-n} \le 0.5 \times 10^{-(N+1)} \le \varepsilon$$

so that $\left|a_n - \sqrt{2}\right| < \varepsilon$ for every $n > N$ as desired.

Usually a proof of convergence requires more algebra than we see here – *but the principles are just the same*:

- Given an $\varepsilon > 0$, we must find N such that $|a_n - L| < \varepsilon$ for every $n > N$.

The figure below gives an idea of the notion of convergence for another sequence. Here the limit is 1 (though we have no *proof* of that fact). The broken lines represent $1 \pm \varepsilon$ for some value of ε. Note that a_8, a_9, a_{10} all lie within the tolerance but $N = 7$ would not work in the proof because a_{11} is outside the strip so that it is *not true* that $|a_n - 1| < \varepsilon$ *for every* $n > N$.



It is apparent that in this case the terms of the sequence do not approach the claimed limit in a regular manner but that they jump around inside gradually smaller bands around the limit. This is another important aspect of convergence: we are concerned with the *eventual* behavior of the sequence rather than any finite set of terms.

Example 1 *The sequence* $(1 + 1/n) \to 1$.

Proof Let $\varepsilon > 0$. We must find N such that

$$\left|1 - \left(1 + \frac{1}{n}\right)\right| < \varepsilon$$

for every $n > N$. Now

$$\left|1 - \left(1 + \frac{1}{n}\right)\right| = \frac{1}{n}$$

and this will be less than ε if $n > 1/\varepsilon$. Therefore if we choose any $N \geq 1/\varepsilon$ it follows for all $n > N$ that

$$\left|1 - \left(1 + \frac{1}{n}\right)\right| = \frac{1}{n} < \frac{1}{N} \leq \frac{1}{1/\varepsilon} = \varepsilon$$

which is precisely what we needed to prove. ■

Example 2 *Prove that* $\lim_{n\to\infty} 1/n^2 = 0$.

Proof Let $\varepsilon > 0$. This time we must find N such that

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \varepsilon$$

for every $n > N$ and this will be valid if $n^2 > 1/\varepsilon$. Therefore if we choose any $N \geq \sqrt{1/\varepsilon}$ it follows for all $n > N$ that

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \frac{1}{N^2} \leq \frac{1}{1/\varepsilon} = \varepsilon$$

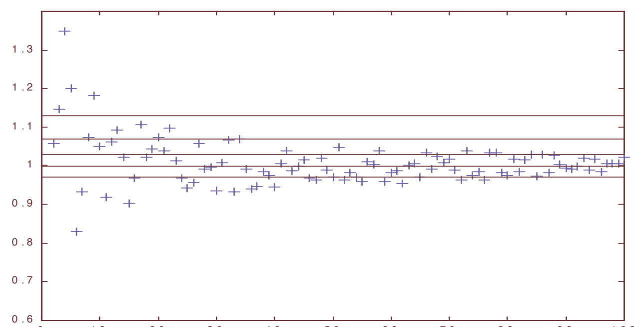
which is precisely what we needed to prove. ■

Note that the basic structure of both these proofs is identical. This approach can be used as a template for many "ε–N" proofs of convergence of sequences.

Note too that for these proofs we need to know in advance the value of the limit we are trying to establish. Sometimes a computer investigation can point us in the right direction.

Note A sequence cannot have two separate limits.

This intuitively obvious statement is easily see to be true by graphical demonstration. In the figure below we illustrate a sequence converging to one limit as its terms eventually all lie within a band about that limit. A possible alternative limit and a similar band about it is seen to contain no terms of the sequence after it settles into the band about the true limit. However close the second "limit" was to the true one we could blow the picture up and choose narrow enough bands to reproduce essentially the same picture.



MATLAB Examples Finding candidate limits

1. Generate a table of the terms a_n for $n = 1, 10, 100, 1000, \ldots, 10^{10}$ for the sequence given by $a_n = \frac{6n-1}{5+3n}$. Use this information to find a candidate for the limit and then prove that this is indeed the limit of this sequence.

```
EDU» for k=1:11
 n=10^(k-1);
```

```
 a=(6*n-1)/(5+3*n);
 fprintf('%15d %12.6f \n',n,a);
end
              1    0.625000
             10    1.685714
            100    1.963934
           1000    1.996339
          10000    1.999633
         100000    1.999963
        1000000    1.999996
       10000000    2.000000
      100000000    2.000000
     1000000000    2.000000
 1.000000e+010    2.000000
```

It appears that 2 is a possible limit. We next prove that $\dfrac{6n-1}{5+3n} \to 2$.

Let $\varepsilon > 0$ be given. Now

$$\left|\frac{6n-1}{5+3n} - 2\right| = \left|\frac{6n-1-2(5+3n)}{5+3n}\right| = \frac{11}{5+3n}$$

which is less than ε if

$$5+3n > \frac{11}{\varepsilon}$$

This condition will certainly be satisfied if $3n > \frac{11}{\varepsilon}$ and so we may choose $N = 11/3\varepsilon$. Then if $n > N$ it follows that

$$\left|\frac{6n-1}{5+3n} - 2\right| = \frac{11}{5+3n} < \frac{11}{3n} < \frac{11}{3(11/3\varepsilon)} = \varepsilon$$

which is precisely what was to be proved. ■

2. Find a candidate limit for the sequence $(n\sin(1/n))$ by tabulating its values for $n = 50 : 50 : 500$

```
EDU» for n=50:50:500
 a=n*sin(1/n);
 fprintf('%6d %15.7f \n',n,a)
end
```

```
 50    0.9999333
100    0.9999833
150    0.9999926
200    0.9999958
250    0.9999973
300    0.9999981
350    0.9999986
400    0.9999990
450    0.9999992
500    0.9999993
```

This evidence suggests that the sequence is converging to a limit somewhere near 1. Does this fit with your intuition? (No? Well, that's one reason why we need to study convergence of sequences carefully.) We will investigate this particular sequence later.

3. Supposing the limit of the sequence above is indeed 1, find values of N such that $|n \sin(1/n) - 1| < \varepsilon$ whenever $n > N$ for each of $\varepsilon = 10^{-2}, 10^{-4}$, and 10^{-6}.

   From the table above it appears that by the time we reach $n = 50$, all terms are already within 10^{-4} of the proposed limit. Such a value of N would also certainly guarantee that a tolerance of 10^{-2} is satisfied but we can try to get a better N for this case. Tabulating the first 20 terms of the sequence and their "errors"

```
EDU» for n=1:20
 a=n*sin(1/n);
 fprintf('%6d %15.7f %12.7f \n',n,a,abs(a-1))
end
  1 0.8414710 0.1585290
  2 0.9588511 0.0411489
  3 0.9815841 0.0184159
  4 0.9896158 0.0103842
  5 0.9933467 0.0066533
  6 0.9953768 0.0046232
  7 0.9966021 0.0033979
  8 0.9973979 0.0026021
  9 0.9979437 0.0020563
 10 0.9983342 0.0016658
 11 0.9986232 0.0013768
 12 0.9988430 0.0011570
 13 0.9990141 0.0009859
```

```
 14 0.9991499 0.0008501
 15 0.9992594 0.0007406
 16 0.9993491 0.0006509
 17 0.9994234 0.0005766
 18 0.9994857 0.0005143
 19 0.9995384 0.0004616
 20 0.9995834 0.0004166
```

we see that the "errors" appear to be decreasing steadily. This suggests that $N = 4$ will suffice for $\varepsilon = 10^{-2}$.

If we assume that the errors are indeed decreasingly continually, then we can use a while loop to determine the other values of N :

```
EDU» n=20; a=n*sin(1/n);
EDU» while abs(a-1)>1e-4
 n=n+1;
 a=n*sin(1/n);
end
EDU» [n,a]
ans =
 41  0.99990085561608
```

so that $N = 40$ appears sufficient for $\varepsilon = 10^{-4}$. Similarly, we obtain $N = 408$ for $\varepsilon = 10^{-6}$.

2.0.2 Exercises

1. Show that the sequence (c) where c is a constant converges to c. That is show (using the definition) that $\lim c = c$.

2. Use the ε–N definition to show that $1/n \to 0$ as $n \to \infty$.

3. Prove each of the following limits:

   a. $\lim \left(2 + \frac{1}{n}\right) = 2$ b. $\lim \dfrac{2n-3}{n+1} = 2$ c. $\lim \dfrac{n^2+3}{2n^2} = \frac{1}{2}$

4. What does it mean to say that a particular sequence does not converge? Try to prove that the sequence (n) does not converge.

5. Use MATLAB to find candidates for limits of the following sequences (if they exist):

   a. $n^2 \left(1 - \cos\left(1/n\right)\right)$ b. $\left(-1\right)^n / n^2$ c. $\dfrac{2^n}{n!}$

6. For each of the sequences in Exercise 5, try to find N such that $|a_n - L| < \varepsilon$ whenever $n > N$ for $\varepsilon = 10^{-4}, 10^{-6}$. Can you *prove* any of these limits?

7. Compute enough terms of the each of the following sequences to decide whether the sequence is (or appears to be) convergent, and, if so, what is its limit.

   a. $a_1 = 2; a_n = \dfrac{a_{n-1} + 4}{2}$ b. $a_1 = 1; a_{n+1} = \dfrac{a_n + 4}{a_n + 1}$

   c. $a_1 = \sqrt{3}; a_{n+1} = \sqrt{3 + a_n}$

8. If a sequence converges, the differences between successive terms must approach 0. Can you prove this fact? We shall see examples shortly which show that the converse of this result is not valid. That is, the differences between successive terms will converge to 0 but the sequence itself will fail to converge. (Can you find such an example?)

9. Let $a, b > 0$ and define two sequences (a_n) and (b_n) by

$$a_1 = a,\ b_1 = b; \quad a_{n+1} = \sqrt{a_n b_n},\ b_{n+1} = \frac{a_n + b_n}{2} \quad (n \geq 1)$$

   For each of the (a, b) pairs $(1, 2), (1, 10), (5, 6), (10, 1000)$ examine the convergence of these sequences. Generate enough terms to decide whether they converge and, if so, what are their limits. What do you notice about the limits of (a_n) and (b_n)? Can you explain your observation?

3 Arithmetic Rules for Limits of Sequences

In the previous section, we saw that the definition can be used to establish limits of various sequences. However, this would be a tedious process if we had to resort to this definition every time we study a new sequence. Fortunately there are a number of rules we can use to simplify this process greatly. These arithmetic rules are obvious for the most part but they do need some proof. Some of these proofs will be included here; the others are essentially similar in nature.

We begin with a summary of these arithmetic rules:

Theorem 1 *(Arithmetic of Limits)*

Suppose that (a_n) and (b_n) are two convergent sequences with limits A and B respectively. Then

1. *Constant Multiple:* $ca_n \to cA$ *for any constant* c

2. *Sums:* $a_n + b_n \to A + B$

3. *Differences:* $a_n - b_n \to A - B$

4. *Products:* $a_n \cdot b_n \to A \cdot B$

5. *Reciprocal:* $1/b_n \to 1/B$ *provided that* b_n, B *are nonzero*

6. *Quotients:* $a_n/b_n \to A/B$ *provided that* b_n, B *are nonzero*

We shall illustrate the proofs of these results with just a few examples.

First, for the sum of two sequences:

Suppose that $a_n \to A, b_n \to B$ and let $\varepsilon > 0$. Now from the definition of convergence it follows that for *every* positive number (including $\varepsilon/2$) we can find appropriate values of N such that each of the sequences is within this tolerance for $n > N$. In particular therefore there are numbers N_1, N_2 such that

$$\begin{aligned} |a_n - A| &< \varepsilon/2 \qquad \text{for every } n > N_1 \text{ and} \\ |b_n - B| &< \varepsilon/2 \qquad \text{for every } n > N_2 \end{aligned}$$

It follows that if $n > \max(N_1, N_2)$ both these inequalities hold and then

$$\begin{aligned} |(a_n + b_n) - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

which was precisely what we needed to prove.

The result for a constant multiple can be proved similarly. Then the rule for differences follows from these two (using a multiplier of -1).

The special case of the product rule where both $|A|, |B| < 1$ is easily proved by a similar argument to that above using the inequalities:

$$\begin{aligned} |a_n b_n - AB| &= |a_n (b_n - B) + B (a_n - A)| \\ &\leq |a_n (b_n - B)| + |B (a_n - A)| \\ &= |a_n| \, |b_n - B| + |B| \, |a_n - A| \\ &< |b_n - B| + |a_n - A| \end{aligned}$$

once n is large enough that $a_n < 1$. (We already have $|B| < 1$.) Each of these can be forced to be less than $\varepsilon/2$ just as in the case for sums.

The reciprocal rule is similarly easy to establish and then the quotient rule follows by combining it with the product rule.

Our primary focus here is not on the proofs of these results (although those are an essential part of a full understanding of this topic) but on their use in establishing limits of other sequences more easily than by using the definition itself.

We have already established that

$$\frac{1}{n} \to 0 \text{ as } n \to \infty$$

and using the product rule (4 in the Theorem above) it follows that

$$\boxed{\frac{1}{n^p} \to 0 \text{ as } n \to \infty}$$

for every positive integer p. This result enables us to deal easily with sequences where the terms are rational functions of n.

Example 3 *Evaluate* $\lim \dfrac{3n^2 - 5n + 1}{2n^2 - n}$.

Solution If we divide numerator and denominator by the highest power of n which appears in this expression, we obtain

$$\frac{3n^2 - 5n + 1}{2n^2 - n} = \frac{3 - 5/n + 1/n^2}{2 - 1/n}$$

and we already know that $1/n, 1/n^2 \to 0$. Using the first three rules for the numerator and denominator of this expression, and then rule 6 for the quotient, we get

$$\frac{3n^2 - 5n + 1}{2n^2 - n} = \frac{3 - 5/n + 1/n^2}{2 - 1/n} \to \frac{3 - 5(0) + 0}{2 - 0} = \frac{3}{2}$$

That is $\lim \dfrac{3n^2 - 5n + 1}{2n^2 - n} = \dfrac{3}{2}$.

We can also use the arithmetic of limits to obtain candidate limits for sequences which are defined recursively. It is important here to observe that if $a_n \to L$, then the sequences given by $(a_{n-1}), (a_{n+1})$ also converge to the same limit. (We can simply adjust the value of N by the appropriate amount.) Thus

$$\boxed{\text{If } \lim a_n = L \text{ then } \lim a_{n+k} = L \text{ for any integer } k}$$

Example 4 *The sequence* (a_n) *is defined recursively by*

$$a_1 = 1, \qquad a_{n+1} = \frac{2a_n + 5}{a_n + 2} \quad (n \geq 1)$$

Assuming it converges, find the limit of this sequence.

Solution If this sequence converges to the limit L, then taking limits on both sides of the recurrence relation, we see that L must satisfy

$$L = \frac{2L + 5}{L + 2}$$

Simplifying, we obtain

$$L^2 + 2L = 2L + 5$$

from which we may deduce that $L = \pm\sqrt{5}$. These are the only possible limits for this sequence. We can actually deduce further that $L = \sqrt{5}$ since $a_1 = 1 > 0$ and, from the recurrence relation, if $a_n > 0$ then $a_{n+1} > 0$ also. It follows that all terms of the sequence are positive and therefore the limit L cannot be negative.

Note that we have not proved that this sequence converges at all. All we have shown is that IF it converges then its limit must be $\sqrt{5}$. But this is a BIG IF.

3.0.3 Exercises

1. Using the arithmetic of limits, evaluate each of the following limits (or, if appropriate, state that it does not exist)

   a. $\lim \dfrac{5n-1}{2n+1}$ b. $\lim \dfrac{2n^3-3n+1}{n^3+n^2}$ c. $\lim \dfrac{5n+1}{n^2+1}$ d. $\lim \dfrac{1+n^2}{1+3n}$

2. Give an "ε–N" proof that $\dfrac{1}{\sqrt{n}} \to 0$.

3. Use the arithmetic of limits *and* the result from Exercise 2 to evaluate the following limits

   a. $\lim \dfrac{2n-\sqrt{n}}{n+\sqrt{n}}$ b. $\lim \dfrac{\sqrt{n}}{n+1}$ c. $\lim \left(2+\dfrac{1}{\sqrt{n}}\right)^5$

4. If $a_n \to L$ (and all the terms $a_n > 0$) then $\sqrt{a_n} \to \sqrt{L}$. (*Extra credit:* Can you prove this result?) Use this fact to find limits for the following sequences (if they exist)

   a. $\left(\sqrt{\dfrac{2n+1}{n+2}}\right)$ b. $\dfrac{\sqrt{n}}{\sqrt{n}+\sqrt{2n+1}}$ c. $\dfrac{2n}{\sqrt{n+1}}$ d. $\dfrac{\sqrt{2n^2+1}}{n}$

5. The sequence (a_n) is given recursively by

$$a_1 = 1; \qquad a_n = \frac{1}{2}\left(a_{n-1} + \frac{3}{a_{n-1}}\right) \text{ for } n > 1$$

   Assuming that this sequence converges, find all possible values for its limit. Use MATLAB to check your answer.

6. Repeat the previous exercise for the sequence given by

$$a_1 = 2; \qquad a_{n+1} = \frac{a_n+2}{a_n+1} \text{ for } n \geq 1$$

7. Let (F_n) represent the Fibonacci sequence defined in Section 1.1 and let (r_n) denote the ratios of its successive terms:

$$r_n = \frac{F_{n+1}}{F_n} \text{ for } n \geq 1$$

   (a) Show that $r_{n+1} = 1 + 1/r_n$

   (b) Assuming that (r_n) converges, use the arithmetic of limits to find its limit.

   (c) Use MATLAB to check your candidate limit from part (b).

4 The Sandwich Theorem

We have already remarked (in Example 7 in the previous section) that if all the terms of a sequence are positive then its limit cannot be negative. This seemingly obvious result is a special case of

Theorem 2 *(Order Rule)*

If $(a_n), (b_n)$ *are convergent sequences with* $a_n < b_n$ *for every* n*, then* $\lim a_n \leq \lim b_n$.

Note that we cannot deduce that $\lim a_n < \lim b_n$ since for example $1/n \to 0$ and $-1/n \to 0$ too.

Proof Suppose for the moment that $\lim a_n > \lim b_n$. We shall show that this implies that (eventually) $a_n > b_n$ which would contradict our hypothesis. This in turn will imply that the assumption $\lim a_n > \lim b_n$ is impossible which establishes the Theorem.

Denote these limits by A, B and choose $\varepsilon = \dfrac{A-B}{2}$. Since the sequences converge to these limits, there exist N_1, N_2 such that

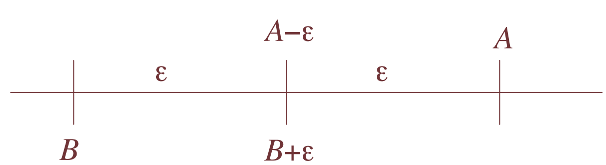
$$\begin{aligned} |a_n - A| &< \varepsilon \text{ for } n > N_1 \\ |b_n - B| &< \varepsilon \text{ for } n > N_2 \end{aligned}$$

Then if $n > \max(N_1, N_2)$, we have

$$B - \varepsilon < b_n < B + \varepsilon, \quad A - \varepsilon < a_n < A + \varepsilon$$

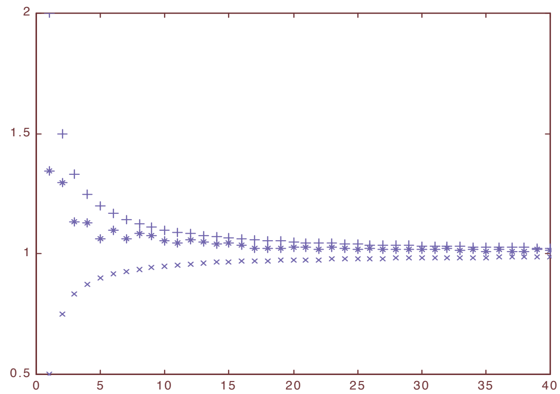
but we chose ε so that $B + \varepsilon = A - \varepsilon$ (see the figure below) from which we deduce that

$$b_n < a_n \text{ for } n > \max(N_1, N_2)$$



This provides the contradiction and completes the proof.■

A closely related result is the Sandwich Theorem which says that if two sequences have the same limit and a third sequences is sandwiched between them then it must have that same limit, too. This is illustrated below with three sequences whose terms are shown with $+, *, \times$ The largest $(+)$ and smallest $(\times)$ sequences have the same limits. The middle one $(*)$ is forced to converge to this same limit too because it is "sandwiched" between the others.



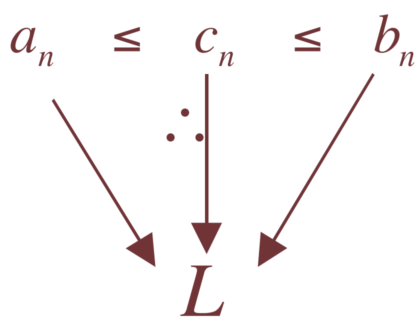
Theorem 3 *(Sandwich Theorem)*
Suppose that $a_n \leq c_n \leq b_n$ *for every* n *and that*

$$\lim a_n = \lim b_n = L$$

Then

$$\lim c_n = L$$

This theorem can be summarized in the following diagram:



Proof Again we can use a simple argument based on the definition of convergence. Let $\varepsilon > 0$. Then there exist N_1, N_2 such that

$$\begin{aligned} L - \varepsilon &< a_n < L + \varepsilon \text{ for } n > N_1 \\ L - \varepsilon &< b_n < L + \varepsilon \text{ for } n > N_2 \end{aligned}$$

In particular, for $n > \max(N_1, N_2)$, we have

$$L - \varepsilon < a_n \leq b_n < L + \varepsilon$$

But c_n lies between a_n and b_n and so we have

$$L - \varepsilon < c_n < L + \varepsilon \text{ for } n > \max(N_1, N_2)$$

which is precisely what we needed to prove.■

The rest of this section is devoted to some very important applications of this Sandwich Theorem.

Example 5 $\lim \frac{1}{2^n} = 0$

Proof Clearly, $0 < \frac{1}{2^n} \leq \frac{1}{n}$ and $\lim 0 = \lim \frac{1}{n} = 0$ and so using the Sandwich Theorem, we deduce that $\lim \frac{1}{2^n} = 0$.

In a similar way we can prove that

$$\boxed{\lim a^n = 0 \text{ for any } |a| < 1}$$

Also, obviously,

$$\lim a^n = 1 \text{ if } a = 1$$

while (a^n) does not converge if $a = -1$ or $|a| > 1$.

Example 6 $\lim 2^{1/n} = 1$

Proof Firstly, for any n, $2^{1/n} > 1$ and so there is a *positive* number $x = 2^{1/n} - 1$. Then by the binomial expansion, we have

$$\begin{aligned} 2 = \left(2^{1/n}\right)^n &= (1+x)^n \\ = 1 + nx + \text{ higher order (positive) terms} \\ &> nx \end{aligned}$$

Therefore $x < 2/n$ and combining all these inequalities, we get

$$1 < 2^{1/n} = 1 + x < 1 + \frac{2}{n}$$

But $1+\frac{2}{n} \to 1$ and so, by the Sandwich Theorem, we deduce that $2^{1/n} \to 1$ as desired.

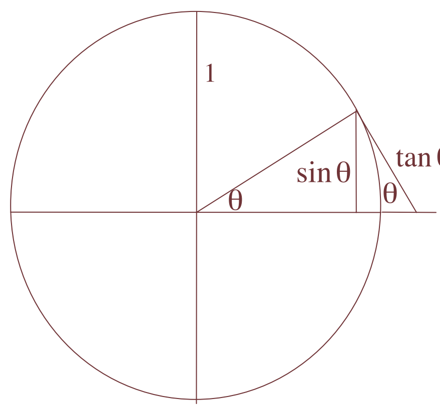
Again this result can be generalized:

$$\boxed{a^{1/n} \to 1 \text{ for any } a > 0}$$

The next three examples establish a very important trigonometric limit, namely

$$\boxed{n \sin \frac{1}{n} \to 1}$$

The following picture will help in all of these:



The critical information which we can gather from this picture is that for any angle $\theta \in (0, \pi/2)$ we have

$$0 < \sin\theta < \theta < \tan\theta$$

since the arc of a unit circle of angle θ has length θ

Example 7 $\lim \sin \frac{1}{n} = 0$

Proof Since $1/n < \pi/2$ for every positive integer n, it follows from the inequality above that

$$0 < \sin \frac{1}{n} < \frac{1}{n} \to 0$$

from which the result follows using the Sandwich Theorem.

Note More generally,

$$\boxed{\sin a_n \to 0 \text{ for every sequence } a_n \to 0}$$

Example 8 $\lim \cos \frac{1}{n} = 1$

Proof Using the half-angle formula $\sin^2\theta/2 = (1-\cos\theta)/2$, the inequality above, and the fact that $\cos\theta \le 1$, we obtain

$$0 \le 1 - \cos\frac{1}{n} = 2\sin^2\frac{1}{2n} < 2\left(\frac{1}{2n}\right)^2 = \frac{1}{2n^2} \to 0$$

It follows from the Sandwich Theorem that $1-\cos\frac{1}{n} \to 0$, or equivalently that $\cos\frac{1}{n} \to 1$.

Note Again there is a more general result:

$$\boxed{\cos a_n \to 1 \text{ for every sequence } a_n \to 0}$$

Example 9 *Finally, we establish that* $\lim n\sin\frac{1}{n} = 1$.

Proof From our basic inequality above we have

$$\sin\frac{1}{n} < \frac{1}{n} < \tan\frac{1}{n} = \frac{\sin(1/n)}{\cos(1/n)}$$

The right hand part of this inequality can be rewritten as

$$\frac{1}{n}\cos(1/n) < \sin(1/n)$$

or

$$\cos(1/n) < n\sin(1/n)$$

Similarly the left hand part can rewritten

$$n\sin(1/n) < 1$$

Together we have

$$\cos(1/n) < n\sin(1/n) < 1$$

and using the Sandwich Theorem, and the previous Example, we deduce that $n\sin(1/n) \to 1$.

Note More generally,

$$\boxed{\frac{\sin a_n}{a_n} \to 1 \text{ for every sequence } a_n \to 0}$$

4.0.4 Exercises

1. Determine whether the following sequences converge. Find the limits for those that do.

   a. $a_n = \left(\frac{-1}{2}\right)^n$ b. $b_n = (1.01)^n$

   c. $c_n = \left(\frac{2^n}{3^n}\right)$ d. $x_n = 1/3^n$

   e. $a_n = \frac{2^{n-1} - 1}{2^n}$ (Hint: divide numerator and denominator by 2^n)

   f. $b_n = \frac{2^n + 3^n}{3^{n+1} + 2^{n+1}}$ (Hint: divide numerator and denominator by 3^n)

2. Use the Sandwich Theorem to evaluate the following limits

   (a) $\lim \frac{1}{n} \cos n$ (Hint: $|\cos n| \leq 1$)

   (b) $\lim \left(\frac{2n}{n+1}\right)^{1/n}$ (Hint: $1 \leq \frac{2n}{n+1} < 2$)

   (c) $\lim (2^n + 3^n)^{1/n}$ (Hint: $1 + x^n < 1 + x$ for $0 < x < 1$)

3. Evaluate the following limits

   a. $\lim \sin \frac{3}{n}$ b. $\lim \cos \frac{2n+1}{n^2}$ c. $\lim \frac{\sin (1/n)}{1/n}$

   d. $\lim n \sin (1/3n)$ e. $\lim \frac{\sin (2/n)}{\sin (3/n)}$ f. $\lim 4^n \sin 4^{-n}$

4. Show that if $a_n \to 0$ then

$$\frac{1 - \cos a_n}{a_n} \to 0$$

5 Application: A Square Root Algorithm

Although your calculator and computer have the square-root function built into their hardware, the question of how these square-roots are evaluated is of some interest. The algorithm we discuss in this section is the basis of many computer algorithms. It is also a good practical application of all we have done on convergence of sequences.

The task of finding the square root of a number a, say is equivalent to solving the equation

$$x^2 = a$$

which is, of course equivalent to

$$x = a/x$$

or, in turn (by adding x to both sides and dividing by 2) to

$$x = \frac{x + a/x}{2}$$

Note that if $0 < x < \sqrt{a}$ then $a/x > \sqrt{a}$ and, similarly, if $x > \sqrt{a}$ then $0 < a/x < \sqrt{a}$. That is, the right-hand side of this last equation forms the average of one quantity bigger than $\sqrt{a}$ and one smaller than $\sqrt{a}$. It is reasonable to hope therefore that $\frac{x + a/x}{2}$ will be closer to $\sqrt{a}$ than either x or a/x.

This suggests the following *iterative* scheme for generating successively better (we hope) approximations to $\sqrt{a}$:

Guess an approximation of $\sqrt{a}$; call it x_0

Define a sequence of (better) approximations by

$$x_{n+1} = \frac{x_n + a/x_n}{2}$$

for $n \geq 0$.

Before discussing the convergence of this sequence, let us consider what happens if we try to find $\sqrt{7}$ using this algorithm with the initial guess $x_0 = 3$. Then we get

$$\begin{aligned} x_1 &= \frac{3 + 7/3}{2} = 2.6667 \\ x_2 &= \frac{2.6667 + 7/2.6667}{2} = 2.6458 \\ x_3 &= \frac{2.6458 + 7/2.6458}{2} = 2.6458 \end{aligned}$$

so that the sequence already appears to have "converged to 4 decimal places". Also note that $\sqrt{7} = 2.6458$ to 4 d.p.

This looks promising. We shall now consider the convergence of this algorithm and then how we could implement this idea in MATLAB.

Claim If the sequence given by $x_0 > 0$ and $x_{n+1} = \frac{x_n + a/x_n}{2}$ converges, then its limit is $\sqrt{a}$.

Proof Suppose that $x_n \to L$ then, by the arithmetic of limits L must satisfy the equation

$$L = \frac{L + a/L}{2}$$

But we have already seen that this equation is equivalent to $L^2 = a$ so that $L = \pm\sqrt{a}$. If $x_0 > 0$ it is easy to see that all the terms of the sequence are positive and so the limit must be $\sqrt{a}$.

- Of course, we have not proved *that* the sequence converges; only that *if* it converges then it has the limit we want.

The next few results will complete the proof that this algorithm indeed converges to $\sqrt{a}$.

Claim If $0 < x_0$, then $x_{n+1} \geq \sqrt{a}$ for $n \geq 0$.

Proof We shall establish that $x_{n+1} - \sqrt{a} \geq 0$. Now, by definition

$$\begin{aligned} x_{n+1} - \sqrt{a} = \frac{x_n + a/x_n}{2} - \sqrt{a} = \frac{1}{2}\left[x_n - 2\sqrt{a} + a/x_n\right] \\ = \frac{1}{2x_n}\left[x_n^2 - 2x_n\sqrt{a} + a\right] \\ = \frac{1}{2x_n}\left(x_n - \sqrt{a}\right)^2 \\ \geq 0 \end{aligned}$$

since $x_n > 0$ and $\left(x_n - \sqrt{a}\right)^2 \geq 0$.

Claim For every $n \geq 1$, $\sqrt{a} \leq x_{n+1} \leq x_n$. That is the sequence (x_n) is decreasing and bounded below by $\sqrt{a}$ for $n \geq 1$.

Proof We already have proved that $\sqrt{a} \leq x_{n+1}$. It is enough therefore to prove that if $n \geq 1$ (so that $x_n \geq \sqrt{a}$) then $x_{n+1} \leq x_n$. Now, $x_n \geq \sqrt{a}$ implies that $a/x_n \leq \sqrt{a} \leq x_n$. The average of two numbers is necessarily smaller than the larger of the two numbers; therefore

$$x_{n+1} = \frac{x_n + a/x_n}{2} \leq x_n$$

There is a theorem which states that if a sequence is decreasing and bounded below then it is convergent. (Basically, the only options for a decreasing sequence are to "go off" to $-\infty$ or to converge to some limit.) With this theorem our task is done since we have proved that

1. *If* (x_n) converges then its limit is $\sqrt{a}$, and
2. (x_n) is decreasing and bounded below (for $n \geq 1$) and therefore convergent.

However, we have not established this result. In the interests of completeness, we shall complete the proof algebraically without using this theorem.

Claim $x_n \to \sqrt{a}$.

Proof We shall only consider $n \geq 1$ for which we already know $\sqrt{a} \leq x_{n+1} \leq x_n$ and therefore $a/x_n \leq a/x_{n+1} \leq \sqrt{a}$. We also know (from the definition) that $x_{n+1} = \dfrac{x_n + a/x_n}{2}$. It follows that

$$\begin{aligned} 0 &\leq x_{n+1} - \sqrt{a} \leq x_{n+1} - a/x_{n+1} \\ &\leq x_{n+1} - a/x_n = \left(x_n - a/x_n\right)/2 \end{aligned}$$

In summary:

$$0 \leq x_{n+1} - \sqrt{a} \leq \frac{1}{2}\,(x_n - a/x_n)$$

and repeating this for $n, n-1, \ldots, 2$ we deduce that

$$0 \leq x_{n+1} - \sqrt{a} \leq \frac{1}{2^n}\,(x_1 - a/x_1) \to 0$$

so that, by the Sandwich Theorem, $x_n \to \sqrt{a}$. ■

MATLAB Examples

1. Perform the first ten iterations of the square root algorithm for finding $\sqrt{7}$ beginning with $x_0 = 3$.

```
EDU» format long g
EDU» a=7;
EDU» x0=3;
EDU» for k=1:10
 x0=(x0+a/x0)/2
end
```

Gives the following sequence of values for x0:

```
x0 =
2.66666666666667
2.64583333333333
2.64575131233596
2.64575131106459
2.64575131106459
2.64575131106459
2.64575131106459
2.64575131106459
2.64575131106459
2.64575131106459
```

It is evident that this sequence has converged *very rapidly* to $\sqrt{7}$ as is evidenced by

```
EDU» sqrt(7)
ans =
 2.64575131106459
```

2. Usually, we would want such an algorithm to stop when successive terms are within some tolerance. For the same example as above with a tolerance 10^{-10} we could use

```
EDU» a=7; x1=3; x0=2;
EDU» while abs(x1-x0)>1e-10
 x0=x1;
 x1=(x0+a/x0)/2;
end
EDU» x1
x1 =
 2.64575131106459
```

   - Note that we need to retain two values in order to test their proximity to each other (and therefore to the solution). We also must ensure that the initial pair `x0, x1` differ by more than the tolerance.

5.0.5 Exercises

1. Implement the square-root algorithm in MATLAB to obtain $\sqrt{a}$ for $a = 2, 3, 11, 17, 2002$, and your alpha code. Use a tolerance of 10^{-10} and compare your answers with the output of the built-in square-root `sqrt`.

2. Write a MATLAB program to generate a table of values of $\sqrt{x}$ for `x=1:0.1:4`.

3. Write a MATLAB function m-file `sqroot.m` which returns the square-root of its argument using tolerance 10^{-12}. Generate a graph of this function and the built-in sqrt for $x \in [0, 10]$. These graphs should be virtually indistinguishable. To get an idea of the errors in your program also plot these errors by using `plot(x,y-sqrt(x))` where `y` is the vector of values from your `sqroot` function.