Let $W_n$ be the “wheel graph of order $n$,” (see \url{http://mathworld.wolfram.com/WheelGraph.html} for a precise definition).

1. What is the smallest $m \in \mathbb{N}$ such that $W_4$ can be embedded (that is, geometrically represented, where vertices are unique points and edges are arcs, and edges meet only at endpoints) in $\mathbb{R}^m$ with each edge a straight line segment of unit length? What about $W_5$, $W_6$, $W_7$? In general?

2. Let $W_n^2$ denote the graph obtained from $W_n$ by adding a single vertex and connecting it to all vertices of $W_n$ except the hub vertex. We also refer to this new vertex as a hub vertex—so $W_n^2$ has two hub vertices. What is the smallest $m \in \mathbb{N}$ such that $W_4^2$ can be embedded in $\mathbb{R}^m$ with each edge a straight line segment of unit length? What about $W_5^2$, $W_6^2$, $W_7^2$? In general?

3. In general, for $k \in \mathbb{N}$, $k > 2$, let $W_n^k$ denote the graph obtained from $W_n^{k-1}$ by joining an additional vertex to all vertices of $W_n^{k-1}$ that are non-hub vertices. What is the smallest $m \in \mathbb{N}$ such that $W_4^k$ can be embedded in $\mathbb{R}^m$ with each edge a straight line segment of unit length? What about $W_5^k$, $W_6^k$, $W_7^k$? In general?

Disclaimers: (a) These were worked on by last year’s group. (b) The project(s) this year, which have yet to be determined, may be more topologically focused than these very geometrically flavored problems. (c) Feel free to submit even if you only worked on a portion of these questions.