The transient response of finite bi-periodic layered structures under axisymmetric loading conditions are analyzed using a double integral transform technique. The effect of the structures of attenuation and propagation zones of the corresponding infinite systems on the transient response is studied. The relation between transformed (frequency and radial wavenumber) and physical (temporal and spatial) variables is emphasized. For verification purposes, a finite element analysis is also performed, and the finite element results are compared with those derived by the double integral transform. It is shown that, in weakly coupled layered systems with narrow propagation zones in the transformed domain, the transient waves are localized close to the circular area where the load is applied. As the coupling between the layers increases, the transmission of stress waves through the layered medium is enhanced, and stress localization diminishes. It is found that weak coupling between layers affects the distribution of the shear stress field more than that of the longitudinal stress field.

1. INTRODUCTION

Investigation of the response of a linear elastic system in the frequency domain provides useful information on its dynamics. The spacing of the natural frequencies provides insight about the steady-state motion of the system. Eigenfunctions corresponding to the natural frequencies are qualitative indicators of the spatial distribution of the steady-state motion. In references [1] and [2], it was examined how the elastic properties of a bi-periodic infinite layered structure affect the locations of the natural frequencies of the corresponding finite layered system. Even though physical arguments based on the analysis in the frequency domain can be very helpful in understanding the basic dynamics of a system, their use is rather limited in explaining a wide range of wave propagation phenomena, such as early-time transient response and near-field wave patterns. This is because wave fields in layered structures are formed by the interaction of pressure (P) and shear (S) waves with mode conversions occurring at the interfaces between layers. The effects of these
interactions are especially predominant in weakly coupled layered structures. Weak coupling causes strong reflected waves, and strong local waves, such as Rayleigh and Stoneley waves; such waves are known to carry most of the energy and, thus, play a crucial role in shaping the wave field (cf. references [3–5]). Thus, studying the local transient wave fields is of practical interest in many engineering applications such as non-destructive testing and the design of composite materials and parts.

In this study, a computational procedure for studying axisymmetric wave propagation in layered bi-periodic structures is developed, and time domain stress analyses of two layered structures with different elastic and geometric properties are performed. The main objective of the computational work is to study the effects of the widths of the propagation zones of the infinite systems on the transient response of the corresponding finite systems. The analysis is based on the numerical inversion of double integral transforms. In references [2] and [6], the Laplace transformation was adopted to transform time to frequency, and the Hankel transform (of order zero and one) to transform the radial spatial co-ordinate to the radial wavenumber. In the inversion process followed, the highly efficient fast Fourier transform algorithm is used to reconstruct the transient response and a numerical inverse Hankel transform is developed to account for the radial dependency of the solutions. The main difficulty in numerically inverting double integral transforms in weakly coupled systems stems from the problem of exponential dichotomy which is inherent in computing transfer functions from the transfer matrix with high spectral condition number. The larger the attenuation zones, the higher the spectral condition number. A historical review of the methods for this problem is provided by reference [7]. In that reference, the loss of precision problem is addressed in detail and a matrix method summarized in reference [8] is adopted to overcome the numerical instability. The numerical method adopted in this work has similarities with the direct global matrix (DGM) approach [9], [10] and the method suggested by Mal [11].

For verification purposes, and extensive finite element analysis is performed. The commercial finite element analysis package ABAQUS® is employed for this purpose. Since the problem under consideration is linear, the implicit time integration (Hilbert–Hughes–Taylor) operator is used without concern about cost and reliability issues involving this operator in non-linear analysis. A detailed description of the technique has been reported [12, 13], and some of its applications can be found in reference [14].

2. FORMULATION OF THE WAVE PROPAGATION PROBLEM

The layered structure under consideration is depicted in Figure 1. The system consists of 10 layers (five bi-periodic sets of two layers). Two different configurations are considered, denoted as system I and system II. The stiff layers of system I are weakly coupled by thin soft layers, whereas in system II there is stronger coupling between the stiff layers. The elastic and geometric properties of the two systems are presented in Appendix A where parameters $\tau_p$ and $\tau_s$ are the ratios of the mechanical impedances of the two materials for pressure (P) and shear (S) waves, respectively, and $v_p$ and $v_s$ are the ratios of the times of travel of pressure (P) and shear (S) waves at phase velocities through the full thickness of each layer (layer A and layer B) respectively. The values of $\tau_p$ and $\tau_s$ control the level of coupling and the strength of mode conversion on the interface between the two layers of a bi-periodic system. The more the values of $\tau_p$ and $\tau_s$ deviate from unity, the weaker the coupling for the associated wave mode becomes. The applied force is axisymmetric, and the footprint of the pressure field is a circular area of 6·35 mm radius (cf. Figure 2). The sensor locations where the stresses are numerically computed are located at radii $r = 0·0, 6·35, 12·7$ and $25·4$ mm.
The transfer matrix formulation for axisymmetric wave propagation in a bi-periodic elastic medium is developed in references [2] and [6]. From this formulation, the transfer functions relating the applied forces to stresses and displacements at any intersection can be readily obtained provided that the $n$th power of of transfer matrix $[T_{m}]$ is computed free from numerical errors due to exponential dichotomy. However, the condition number of the transfer matrix $[T_{m}]$ is so large (especially for system I) that the floating point arithmetic (finite precision) quickly becomes insufficient to provide acceptable numerical results. Thus, calculating the transfer functions with direct multiplications of transfer matrices is of rather limited practical use. This numerical instability problem prevails itself even more severely for the case of weakly coupled systems as in system I due to thin propagation and large attenuation zones in the integration domain $(k, \omega)$. As a result, the eigenvalues of the transfer matrix $[T_{m}]$ become very large, and the condition number becomes even larger due to the reciprocal eigenvalues corresponding to incoming and outgoing waves. In geometrical terms, the directions of the eigenvalues defining the basis of the vector space of the displacement and stress state become so close to each other that the precision of the floating point arithmetic may not be able to capture the difference. Consequently, a null space is artificially introduced in the computations.

The advantages and disadvantages of various approaches in dealing with the exponential dichotomy problem have been discussed in the literature. It appears that the problem was first observed by Dunkin [15] in the seismic wave propagation context. The technique available to avoid the precision problem are examined in reference [16]. The numerical
method adopted here has similarities with the direct global matrix (DGM) approach (cf. references [9, 10]) and the method suggested in reference [11]. As shown below, this method is simple to implement and numerically reliable for a large domain in the \((k, \omega)\) plane.

Once the transfer functions are computed free of numerical errors in the integration domain \((k, \omega)\), the fast Fourier transform (FFT) procedure is employed to integrate the frequency response function in the frequency co-ordinate at fixed wavenumber values leading to a set of so-called “wave modes”. A Hankel transform procedure is then used to compute the responses at four epicentral distances \((r = 0\cdot0, 6\cdot35, 12\cdot7 \text{ and } 25\cdot4 \text{ mm})\) by integrating the wave modes corresponding to fixed values of the wavenumber. In what follows, a procedure for error-free sampling of the transfer functions in the integration domain is developed.

The elastodynamic states on both faces of a bi-periodic set of two layers (cf. Figure 1) are related by the transfer matrix \([T_{set}]\) (cf. Appendix B for the entries of the transfer matrix of a generic layer):

\[
\begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{i+1} = [T_{set}] \begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{i},
\]

(1)

where the elements of the transformed displacement \((\vec{u}_{i}^{1} \text{ and } \vec{u}_{i}^{0})\) and stress \((\vec{\sigma}_{i}^{1} \text{ and } \vec{\sigma}_{i}^{0})\) vectors at interface \(i\) are given as

\[
\{d_{L}\}_{i} = \begin{pmatrix} \vec{u}_{i}^{1} \\ \vec{u}_{i}^{0} \end{pmatrix}, \quad \{f_{L}\}_{i} = \begin{pmatrix} \vec{\sigma}_{i}^{1} \\ \vec{\sigma}_{i}^{0} \end{pmatrix},
\]

and the transfer matrix \([T_{set}]\) is obtained in analytical form in reference [6]. In the above expressions, the overbar and the superscript 0 and 1 denote, respectively, Laplace transform with respect to the scaled time, \(\tau\), and the Hankel transforms of order 0 and 1 with respect to the scaled radial co-ordinate, \(\rho\). For a layered structure with \(N\) periodic sets, the corresponding relation between the stresses and displacements on the top and the bottom surfaces of the structure becomes

\[
\begin{pmatrix} d_{L} \\ -f_{L} \end{pmatrix}_{N} = \begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{N+1} = [T_{set}]^{N} \begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{1},
\]

(2)

where the vector \([f_{L}]_{1}\) denotes the applied force, and \([f_{L}]_{N} = \{0\}\) for a traction-free bottom surface. In theory, the resulting matrix equation can be solved for \([d_{L}]_{1}\) and \([d_{L}]_{N}\). Once this operation is accomplished, the displacement and force vectors at the \(m\)th interface between \(m\) and \(m+1\) periodic sets can be obtained:

\[
\begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{m} = [T_{set}]^{m-1} \begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{1}.
\]

(3)

This computation requires matrix multiplications which cause the numerical instability in the resulting transfer matrices (cf. reference [8]), and an alternative computational approach must be followed.

Axisymmetric harmonic waves propagating through the layered system of infinite axial and radial extent can be studied using the notion of propagation constant, \(\mu\), which relates the state vectors at the two boundaries of a set by the relation

\[
\begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{j+1} = e^{\mu} \begin{pmatrix} d_{L} \\ f_{L} \end{pmatrix}_{j}.
\]

(4)
Using equation (4) and the transfer matrix \([T_{tr}]\), the dispersion relation for the stress wave propagation in the bi-periodic elastic medium is obtained as \([2, 6]\)

\[
\cosh (\mu) = - \frac{R_1}{4} \pm \frac{1}{2} \left( \frac{R_1}{4} - R_2 + 2 \right),
\]

(5)

where

\[
R_1 = - \text{Trace} (T_{tr}),
\]

\[
R_2 = \frac{1}{\text{Det} [t_{ij}]} \left( (x_1^2 - 2x_2x_3 + x_4) + \text{Det} \left[ - [t_{ij}]^{-1} [t_{ij}] - [t_{ij}] [t_{ij}]^{-1} \right] \right),
\]

(6)

and where the operator \(\text{Trace}\) denotes the trace of a matrix, the matrix \([T_{tr}]\) is defined by relations (21), \([t_{ij}]\) denotes the \(ij\)th \((2 \times 2)\) submatrices of \([T_{tr}]\), and \(x_i, i = 1, 2, 3, 4\) are the four elements of the matrix

\[
[t_{ij}]^{-1} = \begin{bmatrix}
 x_1 & x_2 \\
 x_3 & x_4
\end{bmatrix}.
\]

(7)

Since only the waves corresponding to the purely imaginary \(\mu\) can propagate, the relation \(\cosh (\mu) = \pm 1\) defines the boundaries between propagating and attenuating waves.

The domains in the frequency \((\omega)\)–radial wavenumber \((k)\) plane in which the propagation constant, \(\mu\), is purely imaginary are called the propagation zones. The exterior of the propagation zones are the attenuation zones. The attenuation and propagation zones of the elastic systems I and II under investigation are depicted in Figures 4 and 5.

To avoid the successive multiplications required in equation (3), the unknown displacement and force vectors on the interfaces are viewed as being parts of a single state vector, and the following matrix formulation based on the DGM approach is followed.

First, the displacement and stress vectors at the interfaces are joined to a single state vector and the applied pressure field is viewed as an independent force vector:

\[
\{ s \} = \{ d_L \} \{ d_L \} \cdots \{ d_L \} \{ f_L \} \cdots \{ f_L \} \{ 0 \} \{ 0 \},
\]

\[
\{ f \} = \{ f_L \} \{ f_L \} \cdots \{ f_L \} \{ 0 \} \{ 0 \},
\]

where \(\{ s \}\) and \(\{ f \}\) denote the new (unknown) state vector and the (known) applied force vector, respectively.

Relating the force and displacement vectors for each periodic set separately and combining them together into a single matrix equation, a new transfer matrix relation is obtained:

\[
[C] \{ s \} = \{ f \},
\]

(8)

where the matrix \([C]\) is defined as

\[
[C] = 
\begin{bmatrix}
 -[T_{12}]^{-1} & [T_{12}]^{-1} & [0] & [0] & [0] & [0] & \cdots & [0] & [0] & [0] \\
 -[T_{21}]^{-1} & [T_{21}]^{-1} & [0] & [0] & [0] & [0] & \cdots & [0] & [0] & [0] \\
 [0] & -[T_{11}] & -[T_{12}] & \{ I \} & [0] & [0] & \cdots & [0] & [0] & [0] \\
 [0] & -[T_{21}] & -[T_{22}] & [0] & \{ I \} & [0] & \cdots & [0] & [0] & [0] \\
 [0] & [0] & [0] & -[T_{11}] & -[T_{21}] & \{ I \} & \cdots & [0] & [0] & [0] \\
 [0] & [0] & [0] & -[T_{21}] & -[T_{22}] & [0] & \{ I \} & \cdots & [0] & [0] \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
 [0] & [0] & [0] & [0] & [0] & [0] & \cdots & -[T_{11}] & -[T_{12}] & \{ I \} \\
 [0] & [0] & [0] & [0] & [0] & [0] & \cdots & -[T_{21}] & -[T_{22}] & [0]
\end{bmatrix},
\]

(9)
and where the entries \([T_e]\) and \([0]\) denote the \((2 \times 2)\) submatrices of the transfer matrix \([T_{se}]\) and the \((2 \times 2)\) zero matrix, respectively. By inverting the matrix \([C]\), \(\{s\} = [C]^{-1}\{\bar{f}\}\), the elements of the state vector \(\{s\}\) can be computed, and the transformed stresses at interface \(k\) are computed in terms of the applied stress component, \((\bar{\sigma}_{zz})_1\), as follows:

\[
(\bar{\sigma}_{zz})_h = ([K](1 + 4(k - 1), 1) + [K](1 + 4(k - 1), 3))(\bar{\sigma}_{zz})_1,
\]

\[
(\bar{\sigma}_{\rho})_h = ([K](2 + 4(k - 1), 1) + [K](2 + 4(k - 1), 3))(\bar{\sigma}_{zz})_1,
\]

where \([K]_{ij}\) denotes the \(ij\)th entry of the matrix \([C]^{-1}\).

Similarly, the elements of the transformed displacement vector at interface \(k\) are calculated as

\[
(\bar{u}_r)_h = ([K](4(k - 1) - 1), 1) + [K](4(k - 1) - 1), 3))(\bar{\sigma}_{zz})_1,
\]

\[
(\bar{u}_0)_h = ([K](4(k - 1), 1) + [K](4(k - 1), 3))(\bar{\sigma}_{zz})_1.
\]

In the above formulation, successive multiplications of the transfer matrix \([T_{se}]\) are avoided by expanding the dimension of the vector space. The inversion of the matrix \([C]\) eliminates the arithmetic operations over very large and very small numbers which is the primary reason for the loss of precision in computing \([T_{se}]^N\). Thus, using this formulation, the transfer functions can be computed in the integration domain without numerical errors for a larger integration domain. In numerical simulations, the inversion of the matrix \([C]\) is performed using Gaussian elimination due to its efficiency.

### 3. INVERSION OF THE DOUBLE INTEGRAL TRANSFORM

In this section, double integral transforms are inverted to obtain the time domain responses of systems I and II. First, the FFT procedure is applied to calculate the wave modes corresponding to several values of the radial wavenumber \(k\). The wave mode at \(k = 0\) corresponds to the one-dimensional wave propagation discussed in reference [1]. The natural frequencies for constant values of wavenumber are closely spaced, and the frequency response function is highly oscillatory in the propagation zones. In order to capture the dynamics, a high number of samples per point in the frequency domain is needed.

As discussed in reference [1], the numerical problem caused by the placing of the poles of the transfer functions can be avoided by including structural (material) damping in the computational simulations. In this work, several small values of hysteresis loss factors, \(\eta\), (defined by \(E = E_o(1 + j\eta)\) where \(E_o\) is the elasticity modulus of the layer material and \(j = \sqrt{-1}\)) were examined to study its effect on both the transformed and temporal-spatial domain. Considering the elastic and geometric properties of the layered structures under investigation, it was found that a value of \(\eta = 2.0 \times 10^{-4}\) avoided the computational problems caused by the numerical evaluations of improper integrals. This value of damping does not cause any noticeable effects on the responses in the time windows considered in the analysis. After obtaining transient responses (wave modes) for fixed values of the radial wavenumber \(k\) by a FFT procedure, the inverse Hankel transforms for each time instant are evaluated numerically.

#### 3.1. COMPONENTS OF THE APPLIED FORCE AND THEIR TRANSFORMS

In the numerical simulations, a unit force in the \(z\)-direction is applied to a circular area on the upper surface of the layered structures (systems I and II). The force is trapezoidal in time, and the radius of the circular area where the pressure field is applied is constant. This loading condition with the magnitude \(\sigma_o\) can be expressed as

\[
(\sigma_o)_h = \sigma_o f_i(r)f_e(t),
\]

(12)
where $f_1(r)$ and $f_2(t)$ represent the spatial (radial) and temporal dependency, respectively, of the applied pressure field. The components $f_1(r)$ and $f_2(t)$ used in the simulations and their relevant transforms are shown in Figures 2 and 3. As will be shown in the following computations, the kernel of the double inverse transforms is separated into two single-variable functions due to the decoupling of the applied pressure field. The integration over the frequency variable, $\omega$, is first performed. Due to separation of variables, the Hankel transform of the function $f_1(r)$ will be treated as constant in the first integration over $\omega$. The structure of $f_1(r)$ plays a role in the second integration over the wavenumber $k$. Moreover, since the sensor location appears in the Bessel function of the double integral transform, it also comes into play in the second integration. In the numerical simulations, the results of the first integration can be used for various functions $f_1(r)$ and sensor locations. For example, the size of the footprint can be changed, and its effect on the wave propagation can be studied without evaluating any more integrals. The wave modes corresponding to each sampled value of the wavenumber are computed with the FFT procedure, and digitally stored. The transient response at a sensor location for $f_1(r)$ is then constructed by computing the second Hankel inversion integral. This process increases the computational efficiency since unnecessary repetitions of computations are avoided.

Using the relevant transforms of the functions $f_1(r)$ and $f_2(t)$, the integration domain in the frequency–radial wavenumber plane is determined. The Nyquist (cut-off) frequency for $f_1(t)$ is approximately the frequency value where the transform of $f_2(t)$ becomes nearly close to zero. The wavenumber value of the Rayleigh mode asymptote at the Nyquist frequency defines an upper bound for the numerical inverse Hankel transform. Depending upon the power spectrum of $f_2(t)$, lower values of the Nyquist frequency can be chosen. In the following computations, the Nyquist frequency in the scaled frequency $v$ is 2.57 which corresponds to $2.1366 \times 10^6$ Hz (cf. Figure 3(b)). The upper boundary $k^*$ for the inverse Hankel transforms in the scaled wavenumber is chosen as 3.14 for system I and 2.00 for system II. The resulting rectangular integration domains for the two systems are marked by dashed lines in Figures 4 and 5. Numerical experiments have demonstrated that these integration domains guarantee convergence of the numerical results.

### 3.2. Numerical Inversion of the Integrals: Transient Responses

As indicated above, the frequency axis is chosen as the first integration co-ordinate, and the number of sampling points in the frequency direction is much larger than that in the wavenumber direction. The following integrals must be evaluated in order to compute the transient stress components at the $i$th interface of the layered system with $\sigma_s = 1$ Pa:

\[
\sigma_{zz}(r, t) = \int_{0}^{\infty} f_1^2(k) k J_0(kr) \left[ \frac{\hat{\mathcal{H}}(\omega) (\mathbb{K}[e + 4(k - 1), 1])}{\mathbb{K}[1]} \right] e^{-j\omega t} d\omega dk
\]

\[
+ \int_{0}^{\infty} f_1^2(k) k J_0(kr) \left[ \frac{\hat{\mathcal{H}}(\omega) (\mathbb{K}[e + 4(k - 1), 3])}{\mathbb{K}[2]} \right] e^{-j\omega t} d\omega dk,
\]

\[
\sigma_{rr}(r, t) = \int_{0}^{\infty} f_1^2(k) k J_1(kr) \left[ \frac{\hat{\mathcal{H}}(\omega) (\mathbb{K}[e + 4(k - 1), 1])}{\mathbb{K}[1]} \right] e^{-j\omega t} d\omega dk
\]

\[
+ \int_{0}^{\infty} f_1^2(k) k J_1(kr) \left[ \frac{\hat{\mathcal{H}}(\omega) (\mathbb{K}[e + 4(k - 1), 3])}{\mathbb{K}[2]} \right] e^{-j\omega t} d\omega dk.
\]
Note that the stress components in unscaled and scaled co-ordinates are related by $H_2 \sigma_{zz} = \sigma_{zz}$ and $H_2 \sigma_{zz} = \sigma_{zz}$, where $H_A$ denotes the thickness of layer $A$ (cf. Appendix A). Following the FFT procedure, at each time instant, the inverse Hankel transform of relevant order is computed to construct the transient response at a sensor location (equations (13)). In the calculation of transient wave fields using transform techniques, inverse integrals, such as the Hankel transform integrals of the present case, are sometimes called wavenumber integrals; their evaluation herein is performed following references [17, 18].

There are a number of approaches for evaluating Fourier integrals, such as asymptotic approximations and series expansions. These techniques are effective for integrals whose integrands can be represented by relatively simple mathematical expressions, and their
validity holds only under certain conditions such as for constructing farfield solutions. In the case of the problem under consideration, the expressions are obtained through multiplication of \((4 \times 4)\) transfer matrices; thus the corresponding algebraic forms are extremely complicated. In addition, evaluation of such algebraic expressions is not straightforward due to the problem of exponential dichotomy. Here, following a numerical procedure similar to the DGM procedure, the highly efficient fast Fourier transform (FFT) procedure is employed to perform the frequency integrations:

\[
\begin{align*}
\sigma_{zz}(k, t, i) = & \int_{-\infty}^{\infty} (\bar{\sigma}^0_{zz}(k, \omega)) e^{-j\omega t} d\omega \\
= & \int_{-\infty}^{\infty} \tilde{f}_{zz}(\omega) [(K)^{ii} + (K)^{ii-1,1}] e^{-j\omega t} d\omega, \\
\sigma_{zz}(k, t, i) = & \int_{-\infty}^{\infty} (\bar{\sigma}^0_{zz}(k, \omega)) e^{-j\omega t} d\omega \\
= & \int_{-\infty}^{\infty} \tilde{f}_{zz}(\omega) [(K)^{ii} + (K)^{ii-1,1}] e^{-j\omega t} d\omega.
\end{align*}
\]

(14)

The plots in Figure 6 depict the wave modes \(\sigma_{zz}(k, t)\) corresponding to specific values of the wavenumber \(k\) \((k = 4Di, i = 0, 1, \ldots, 7\) and \(\Delta k = 3\pi/30)\) on the second interface for system I, obtained by evaluating equations (14) via the FFT procedure. The plots in Figures 7 and 8 depict the wave modes \(\sigma_{zz}(k, t)\) and \(\sigma_{zz}(k, t)\), respectively, on all four interfaces in superposed form. The inverse Fourier transform is performed using the routines FFTCB and DFFTCB from the Fortran subroutine Library IMSL (reference [19]) on a Cray X-MP at the National Center for Supercomputing Applications at the University of Illinois at Champaign-Urbana. The number of sampling points in \(\omega\) is often chosen to be as high as 70 000. In the Fourier analysis, it is kept in mind that the FFT procedure takes approximately \(N \log (N)\) arithmetic operations, thus, it is very efficient when the number of sampling points \(N\) is a product of small prime factors. The CPU time spent for a time response history at a fixed \(k\) value varies from approximately 60 s to 90 s for the number of sampling points between 50 000 and 70 000. The corresponding wave modes for system II are shown in Figures 9 and 10.

The response history corresponding to \(k = 0\) is the solution of the one-dimensional wave motion (uniform applied force in the radial direction \(r\)) (cf. reference [1] for the one-dimensional analysis). Even though the wave modes corresponding to specific wavenumbers are limited to the response at that wavenumber, some overall remarks regarding the comparison of the two systems can be made. From Figures 7 and 8, it is apparent that at the first interface, system I (whose \(\tau_T\) value is 10 times larger than that of system II) experiences stronger pressure wave reflections than system II. However, for higher values of \(k\), the difference between reflection patterns of the two systems is less visible. It can also be seen that system II induces more shear stress at the interfaces than system I. At \(k = 0\), where the applied pressure field extends to infinity in the radial direction, the value of induced shear stress is zero since no shear load is applied. In computations, this physical fact is observed for both systems. Finally, the choice of maximum values of wavenumber for sampling are justified from Figures 7–10 (cf. reference [6] for the complete sets of time responses at all the interfaces of systems I and II).
The construction of the transient response at a sensor location requires the computation of the inverse Hankel transformation. The stress components are obtained by evaluating the integrals

\[
(\sigma_{zz}(r, t)) = \int_0^{\infty} f_i(k)(\sigma_{zz}^{s}(k, t))kJ_0(rk) \, dk,
\]

\[
(\sigma_{zz}(r, t)) = \int_0^{\infty} f_i(k)(\sigma_{zz}^{s}(k, t))kJ_1(rk) \, dk.
\] (15)

For computational purposes, the upper bounds of the inversions (15) can be approximated by a finite value since for increasing values of \(k\) the maximum amplitude of the time modes decreases. Thus, the following expressions hold:

\[
(\sigma_{zz}(r, t)) \approx \int_0^{N-1} f_i(k)(\sigma_{zz}^{s}(k, t))kJ_0(rk) \, dk,
\]

\[
(\sigma_{zz}(r, t)) \approx \int_0^{N-1} f_i(k)(\sigma_{zz}^{s}(k, t))kJ_1(rk) \, dk.
\] (16)

where \(N\) is the total number of sampling points in the radial wavenumber direction, and the step size is equal to \(\Delta k = k^*/(N - 1)\), where \(k^*\) is the upper bound for the integration in the wavenumber axis. As noted earlier, the upper bound \(k^*\) is determined by either the Hankel transform of the spatial component of the applied force, or the wavenumber determined by considering the Nyquist frequency and the Rayleigh mode asymptote. The least of these two values determines the upper bound in \(k\) for the Hankel inversions.
Reconstruction of a time history requires the evaluation of the integrals (15) at each time instant. The plots in Figure 11 depict the integrand of the first equation (16) for $\sigma_{zz}$ on the second interface of system I at specific time instants. The plots in Figure 12 depict this integrand on all four interfaces of system I in superposed form. Figures 12–15 indicate that the kernels of the wave integral for both systems I and II are dying out with increasing values of wavenumber ($k$). This inspection validates the choices for the upper bounds of the wavenumbers $k^*$. Since the kernels are continuous, for a given wavenumber interval, fitting a polynomial curve for the kernels $f_i(k)(\sigma_{zz}(k, t))k$ and $f_i(k)(\sigma_{zz}(k, t))k$ yields

Figure 6. Inverse Fourier transform of the stress component $\sigma_{zz}$ for system I at fixed values of the wavenumber $k$ on the second interface of a five-set layered structure ($\sigma_0 = 1.0$ Pa). (a) $k = 0$; (b) $k = 0.2$; (c) $k = 0.41$; (d) $k = 0.62$; (e) $k = 0.83$; (f) $k = 1.04$; (g) $k = 1.25$; (h) $k = 1.46$; (i) $k = 1.67$; (j) $k = 1.88$; (k) $k = 2.09$; (l) $k = 2.3$; (m) $k = 2.51$; (n) $k = 2.72$; (o) $k = 2.93$. 
Figure 7. Inverse Fourier transform of the stress component $\sigma_z$ for system I at various values of the wavenumber $k \left( k = 4.4ki, i = 0, 1, \ldots, 7 \right)$ and $\Delta k = \pi/30$ on the four interfaces of the five-set layered structure. Note that the amplitude of the system response decreases with increasing $k$, and that the response for $k = 0$ corresponds to one-dimensional case ($\sigma_z = 100$ Pa). (a) Interface 2; (b) interface 3; (c) interface 4; (d) interface 5.

a rather accurate approximation for the wave integrals. The summation of the integrals over sub-intervals provides the response at the given time instant.

To evaluate efficiently the definite integrals (16) which are approximations of equations (15), a numerical procedure based on fitting low order polynomials for the slowly varying kernels $f^1_i(k)(\sigma^+_z(k,t))i/k$ and $f^1_i(k)(\sigma^-_z(k,t))i$ is introduced. For the sake of generality, the kernel is denoted a function $F(k)$ in $k$. The integration interval $(a, b)$ in $k$ is first partitioned into $(N - 1)$ sub-intervals of length $\Delta k$, where the $j$th sub-interval corresponds to $\left((j - 1)\Delta k + a, j\Delta k + b\right)$. The value of the integral $I_j$ over the $j$th sub-interval is computed by using the fitted polynomial. The value of this generic integral over the interval $(a, b)$ is then obtained by summing the values of integrals over sub-intervals. This procedure can be summarized as follows:

$$I = \int_a^b F(k)J_\nu(rk) \, dk = \sum_{j=1}^{N-1} \int_{(j-1)\Delta k + a}^{j\Delta k + a} F(k)J_\nu(rk) \, dk = \sum_{j=1}^{N-1} I_j, \quad (17)$$

where the function $F(k)$ in $k$ denotes the integrands $(\sigma_z(k,t))i/kf^1_i(k)$ for $\sigma_z$ and $(\sigma_z(k,t))i/kf^1_i(k)$ for $\sigma_z$. The lower limit for the integration is $k = 0$. Thus, the lower bound becomes $a = 0$, and the $j$th sub-interval is $\left((j - 1)\Delta k, j\Delta k\right]$. For simplicity, a linear spline is fitted, i.e., the function $F(k)$ is approximated as $F(k) = z_j k + \beta_j$, where $z_j = F_{j+1} - F_j/\Delta k$ and $\beta_j = -F_{j+1} - F_j(j - 1) + F_j$. $F_j$ and $F_{j+1}$ are $F((j - 1)\Delta k)$ and $F(j\Delta k)$, respectively.
The spatial variable $r$ which denotes the sensor location in the argument of Bessel functions serves as a frequency-like term of the integrand. In order to determine the number of sampling points in these numerical integrations effectively, it is advantageous to keep the zeros of the Bessel function fixed. This can be implemented by carrying the
Figure 10. Inverse Fourier transform of the stress component \( \sigma_z \) for system II at various values of the wavenumber \( k \) \((k = 4\Delta k, i = 0, 1, \ldots, 7\) and \( \Delta k = \pi/30 \)) on the four interfaces of the five-set layered structure (\( \sigma_0 = 1.0 \text{ Pa} \)). (a) Interface 2; (b) interface 3; (c) interface 4; (d) interface 5.

variable \( r \) to the bounds of integration by the change of variables, \( y = rk, \Delta k = (1/r) \text{ dy} \).

This yields the following equivalent expression for \( I_j \):

\[
I_j = \frac{2}{r} \int_{\gamma_1}^{y_j} y J_n(y) \text{ dy} + \frac{\beta_j}{r} \int_{y_j}^{\gamma_2} J_n(y) \text{ dy},
\]

where \( \gamma_1 = ((j - 1)\Delta k)r \) and \( \gamma_2 = (j\Delta k)r \). The integrals in equation (18), \( \int_{\gamma_1}^{\gamma_2} y^m J_n(y) \text{ dy} \) for \( m = 0 \) and 1 are called "moment terms". The moment terms computed for one integral can be stored and reused in additional integrals on different interfaces and sensor locations; this increases the computational efficiency of the procedure. Taking into account the previous spline approximations, integral (17) assumes the form

\[
I = \frac{1}{r} \left( 1 - \frac{1}{r} \sum_{j=1}^{N-1} \beta_j C_j + \beta D \right),
\]

where, \( C_j = \int_{\gamma_1}^{\gamma_j} y J_n(y) \text{ dy} \) and \( D = \int_{\gamma_1}^{\gamma_2} J_n(y) \text{ dy} \).

In constructing the transient responses at points relatively far from the symmetry line, the silent region defined as the time duration from time zero up to arrival of the wavefront is formed by superpositions of waves in a continuous spectrum of the wavenumber. The wave modes corresponding to sampled radial wavenumbers travel at the phase velocities of each layer, however, for a loading which is applied to a constant radius footprint, the velocity of the wave is governed by the dispersion relation and the arrival time at a point which is far from the symmetry axis is formed by the interactions of incoming and reflected waves. Thus, at a relatively far sensor location, the accuracy of the wave integration plays an important role. In order to obtain the silent region accurately
at such sensor locations, a higher order polynomial should be used as the integration spline. For example, the spline $F(k) = a_kk^3 + b_kk^2 + c_kk + d_k$ results in the following form of the integral (17):

$$
I = \sum_{j=1}^{N-1} \frac{1}{r^3} a_j \int_{(j-1)dkr}^{dkr} y^3 J_n(y) \, dy + \frac{1}{r^2} b_j \int_{(j-1)dkr}^{dkr} y^2 J_n(y) \, dy
$$

$$
+ \frac{1}{r} c_j \int_{(j-1)dkr}^{dkr} y J_n(y) \, dy + d_j \int_{(j-1)dkr}^{dkr} J_n(y) \, dy.
$$

(20)

In this approximation, higher order moment terms are present. While better accuracy for integration is obtained, the amount of computation is increased due to additional moment integrals in equation (20). Applying this procedure to perform the inverse Hankel

![Figure 11](image.png)

Figure 11. Integrand of the wave integral (inverse Hankel transform of order zero) of \(\sigma_z\) for system I at various time instants on the second interface (\(\sigma_0 = 1\) Pa). (a) \(t = 0\) ms; (b) \(t = 8\) 4; (c) \(t = 16\) 8; (d) \(t = 25\) 2; (e) \(t = 33\) 6; (f) \(t = 42\) 0; (g) \(t = 50\) 4; (h) \(t = 58\) 8; (i) \(t = 67\) 2; (j) \(t = 75\) 9; (k) \(t = 84\) 0; (l) \(t = 92\) 4; (m) \(t = 100\) 8; (n) \(t = 109\) 2; (o) \(t = 117\) 6.
transforms of orders zero and one, the transient responses at four sensor locations on four interfaces were computed. The construction of the time domain results are performed via a MATHEMATICA® [20] function which is developed for evaluating the inverse Hankel transform using the previously described spline approximations. While, in most of the

Figure 12. Integrand of the wave integral (inverse Hankel transform of order zero) of $\sigma_z$ for system I at various time instants ($t = \Delta t_i$, $i = 0, 1, \ldots, 15$, $\Delta t = 8.4$ $\mu$s) on the four interfaces. Note that the amplitudes of all integrands decrease with time ($\sigma_0 = 1.0$ Pa). (a) Interface 2; (b) interface 3; (c) interface 4; (d) interface 5.

Figure 13. Integrand of the wave integral (inverse Hankel transform of order one) of $\sigma_z$ for system I at various time instants ($t = \Delta t_i$, $i = 0, 1, \ldots, 15$, $\Delta t = 8.4$ $\mu$s) on the four interfaces ($\sigma_0 = 1.0$ Pa). (a) Interface 2; (b) interface 3; (c) interface 4; (d) interface 5.
Figure 14. Integrand of the wave integral (inverse Hankel transform of order zero) of $\sigma_r$ for system II at various time instants ($t = Dt_i$, $i = 0, 1, \ldots, 15$, $Dt = 8.4$ ms) on the four interfaces ($\sigma_0 = 1.0$ Pa). (a) Interface 2; (b) interface 3; (c) interface 4; (d) interface 5.

Transient response integrations, the line spline elements were used, the quadratic spline elements were employed in specific cases, namely for computing the transient response in the first interface at $r = 25.4$ mm. Accurate results were obtained by considering higher

Figure 15. Integrand of the wave integral (inverse Hankel transform of order one) of $\sigma_r$ for system II at various time instants ($t = Dt_i$, $i = 0, 1, \ldots, 15$, $Dt = 8.4$ ms) on the four interfaces ($\sigma_0 = 1.0$ Pa). (a) Interface 2; (b) interface 3; (c) interface 4; (d) interface 5.
order splines for the integrands since the superposition of the wave modes was better approximated. The transient responses of systems I and II at four sensor locations \( r = 0.0, 6.35, 12.7 \) and 25.4 mm on the second interface are depicted in Figures 16–19 for \( \sigma_{zz} \) and \( \sigma_{zr} \) (cf. reference [6] for a closer examination of all the integrands). The accuracy obtained in computing the silent region is a positive indicator for the correctness of the

Figure 16. Comparison of finite element results (————) with the integral transform technique results (———) of the stress \( \sigma_{zz} \) for system I at \( r = 0.0, 0.25, 0.5, \) and 1.0 in \( (r = 0.0, 6.35, 12.7 \) and 25.4 mm) on the second interface \( (\sigma_0 = 1.0 \text{ Pa}) \). (a) \( r = 0 \) in; (b) \( r = 0.25 \) in.; (c) \( r = 0.5 \) in.; (d) \( r = 1 \) in.

Figure 17. Comparison of finite element results (————) with the integral transform technique results (———) of the stress \( \sigma_{zr} \) for system I at \( r = 0.0, 0.25, 0.5 \) and 1.0 in \( (r = 0.0, 6.35, 12.7 \) and 25.4 mm) on the second interface \( (\sigma_0 = 1.0 \text{ Pa}) \). (a) \( r = 0 \) in; (b) \( r = 0.25 \) in.; (c) \( r = 0.5 \) in.; (d) \( r = 1 \) in.
computations. In addition, the phase velocities of the layers calculated from the arrival times of the wavefront at the sensor locations are physically meaningful.

In Figures 7–10, the one-dimensional stress propagation at various fixed values of $k$ is depicted. No radial dispersion exists in these plots. Note the relatively strong
secondary reflections appearing the the one-dimensional transient stress waves, generated at interfaces between layers. Comparing these results to the axisymmetric waves depicted in Figures 16–19, it is observed that in the presence of radial dispersion no strong reflections in the transient waves exist in later times: i.e., the integration of the wave modes to produce the axisymmetric transient response eliminates by interference the strong reflections of the one-dimensional wave modes. This lack of strong secondary waves in the axisymmetric case is attributed to the additional “freedom” of the waves to propagate in the radial direction (radial dispersion of the stress pulse).

4. VERIFICATION OF RESULTS: FINITE ELEMENT ANALYSIS OF AXISYMMETRIC WAVE PROPAGATION IN LAYERED STRUCTURES

For verification purposes, an extensive finite element analysis was also performed. The commercial finite element analysis package ABAQUS® was employed for this purpose. Since the problem under consideration is linear, the implicit time integration (Hilbert–Hughes–Taylor) operator is used without concerns about its cost and reliability in non-linear analysis. The principal advantage of the implicit operator is that the technique is unconditionally stable in time. Regarding the stability of the computations, there is no mathematical limit on the size of the time increment used in the finite element analysis. A more detailed description of the technique is reported in references [12, 13], and some of its applications can be found in reference [14].

The layered structures under consideration extend to infinity in the radial direction. This configuration implies that no reflection from the radial direction can occur. In order to satisfy this no-reflection condition in a finite element analysis, energy absorbing infinite elements (silent boundaries) are employed. Silent boundaries used in ABAQUS® (reference [21]) are implemented in reference [22]. These infinite elements produce very accurate results for low amplitude stress wave propagation; however, they are relatively poor in handling higher frequencies. Also, it is observed that surface waves such as Rayleigh waves are not absorbed to a reasonable degree (cf. reference [22]). This point is especially important for the structures considered herein since the Rayleigh mode of wave motion plays a crucial role due to the slender nature of the geometry. By selecting the location of the silent elements, the effect of erroneous reflections is avoided for the early times.

In this section, results of a finite element analysis performed for systems I and II are compared with those of the integral transform technique. Two major computational issues concerning the transient finite element analysis are addressed, namely, node-to-node oscillation and the performance of silent boundaries. In order to overcome the problem of node-to-node oscillation (numerical reflections from the node boundaries), an intensive convergence study is carried out. It is found that the results from both techniques are generally in good agreement although, in some cases, some diversion is observed in the calculations as time progresses.

4.1. FINITE ELEMENT MODEL

A layered structure consisting of a number of bi-periodic sets has a geometry which is relatively easy to discretize into a finite element mesh. The geometry can be represented by the thicknesses \( H_a \) and \( H_b \) of two layers in the bi-periodic set and the number of bi-periodic sets. The structure extends to infinity in the radial direction. This requires the determination of the location of the boundary between the mesh and the energy absorbing elements. For symmetry of the wave field, the boundary of the silent elements is placed
at a constant distance from the symmetry axis. The boundary is parallel to the symmetry axis.

The pressure field is applied to a circular area with radius 0.5 in. As a result, the resultant wave field is symmetric. In mesh generation, only axisymmetric elements are employed. The area in the wave field is constructed by using the axisymmetric four-node element CAX4R from the element library of ABAQUS while the no-reflection (silent) boundaries are modelled by the four-node infinite element CINAX4.

The convergence of the results are shown by using two finite element meshes, namely, meshes I and II. The distance from the symmetry axis to the silent boundaries is chosen as 76.2 mm so that the reflections from the silent boundaries do not interfere with the incoming waves in the first 25 μs after the arrival of the first wavefront. In mesh I, the number of elements used in the radial direction (including the silent boundary) is 100. For layer A, 15 elements in the z-direction are used. The corresponding aspect ratio is approximately 0.9. Layer B consists of three elements in the z-direction. Thus, the aspect ratio for layer B in mesh I is approximately 9.10. For the second mesh, mesh II, the number of elements in the radial directions is increased to 140. For mesh II, the numbers of elements in the z-direction for layers A and B are 20 and 5, respectively. The aspect ratios for layers A and B are 0.86 and 10.79, respectively.

The total number of elements for mesh I is 8900 and the total number of nodal points is 9100. The maximum number of elements considered by the wavefront solver is 204, and the average number (r.m.s.) during the analysis is 202. The CPU time used for the analysis is 11 059 s for system I and 15 102 s for system III on an IBM RS 6000. The total number of time steps for a 100 μs analysis is 175 for system I, and 203 for system II.

The total number of elements for mesh II is 17 375, and the total number of nodal points is 17 640. The maximum number of elements considered by the wavefront solver is 284 and the average number (r.m.s.) during the analysis is 282. The CPU time used for the analysis is 8613 s for system I and 12 192 s for system II on an IBM RS 6000. The total number of time steps for the 100 μs analysis is 116 for system I, and 169 for system II.

Prior to these two mesh structures, various coarser discretizations were considered, and a smooth convergence in time transients responses was observed as the preliminary meshes converged to mesh I. Even though it was conclusive enough for convergence purposes, mesh II, which is finer than mesh I, was employed in the analysis. An excellent convergence between the results was obtained. In the following sections, only the transient responses obtained using mesh II are presented.

4.2. FINITE ELEMENT RESULTS AND COMPARISON WITH THE INTEGRAL TRANSFORM ANALYSIS

As in any large-scale computational study, a reliability issue arises for the wave propagation problem under consideration. In the computations required for the integral transform technique, approximations were made in evaluating the wave equations which can potentially result in more than an acceptable level of errors. In addition, purely computational errors (numerical instability) in the computations of the transfer functions may exist. On the other hand, the finite element analysis has its own problems, especially in the transient analysis. For example, the existence of numerical reflections from element boundaries (node-to-node resonance) may lead to erroneous results, and inaccuracies due to the fact that the silent elements are inherently ineffective for transmitting surface waves. Thus, a successful comparison between the double integral transform and finite element results is necessary.

The total analysis time of the finite element simulations is 100 μs. Even though the Hilbert–Hughes–Taylor implicit integration operator is unconditionally stable regardless
of the step size of the time increment for the linear problem, the accuracy of the transient results relies on selection of appropriate time steps. The finite element package ABAQUS\textsuperscript{8} allows the use of an automatic time incrementation scheme. In the simulations, the initial step size of the time increment is equal to 0.02\,m\,s, the maximum step size of the time increment is 1.0\,m\,s, and the minimum step size of the time increment is 0.01\,m\,s. It was observed that, during the computations, ABAQUS\textsuperscript{8} tends to increase the initial step size, and never reaches the minimum step size. This proves that the choice for the initial step size is appropriate.

For the finite element results presented here, an artificial damping coefficient $\alpha = -0.16$ is used. The minimum value of this coefficient available from this integration operator is $\alpha = -1/3$. The artificial damping reduces the effects of node-to-node oscillations in the transient responses. As suggested in the ABAQUS\textsuperscript{8} manual, a number of finite element simulation runs were performed to determine the value of the artificial damping coefficient $\alpha$ to eliminate the numerical oscillations in the time responses. This numerical problem is encountered for $\alpha = 0.0$, especially for the shear stress component. As a result, various values of $\alpha$ are considered. For the $\alpha$ values between $-0.16$ and $-0.33$, no further reduction in this oscillatory behavior is observed, thus $\alpha = -0.16$ is used throughout the finite element analysis.

A comparison between the finite element and double integral transform results is shown in Figures 16–19. In general, the transient results from the finite element analysis and the integral transform technique match better for early times (before $t = 50\,m\,s$) than for later times (after $t = 50\,m\,s$). As discussed above, the integral transform results are valid up to a certain time depending upon the step size of the wavenumber ($k$) increment. In the early times, the maximum stress values for $\sigma_{zz}$ of the integral transform results are generally approximately 5\% larger than those from finite element analysis. This difference for the maximum values of $\sigma_{rr}$ sometimes reaches 10\%. However, for both components of the stress tensor, the profile of the incoming waves are quite similar in both sets of results. For the stress $\sigma_{rr}$, the finite element result is more oscillatory. A possible reason for this behavior is that the artificial reflections from the element boundaries, known as node-to-node resonance, are more effective for the shear stress $\sigma_{rr}$ than for $\sigma_{zz}$. In the finite element analysis, the shear stress $\sigma_{rr}$ responses at the epicenter, namely, at the symmetry axis were found to have non-zero values. The maximum values of $\sigma_{rr}$ at the epicenter reached 10\% of the response at $r = 6.35\,mm$ in certain cases. Due to the axisymmetry of the problem, the shear stress $\sigma_{rr}$ is zero at the epicenter. This inconsistency is considered to be a numerical error in computing the stress components. The transfer functions (the frequency responses functions) for $\sigma_{rr}$ at the epicenter using the integral transform technique, and consequently the transient responses for $\sigma_{rr}$ are identical to zero at each interface.

As the interface number increases, the agreement between the two results improves for the early times. For $\sigma_{zz}$, a phase difference becomes apparent with time, while the amplitudes do not differ much. A possible reason for this betterment is that in the finite element analysis, the applied pressure load on the surface is approximated by a set of point loads applied to some nodes in the footprint. The pressure load is applied to the nodal points as a superposition of local point loads. This approximation introduces errors in the local wave field. However, away from the nodes at which the pressure load is applied, the effect of point loads at nodal points diminishes.

For the stress component $\sigma_{rr}$, in the early times, a reasonable agreement between results is obtained. Unfortunately, starting at about 40\,\mu s, a large difference between the two solutions is observed. While the transient results from the integral transform technique tend to diminish after 40\,\mu s, in the finite element results strong reflections, in some cases
as large as the wave front, dominate the response at the sensor point. This effect is more visible in system II. The layered structure extends to infinity in the radial direction, therefore the existence of such strong reflections after 40 µs cannot be explained. As noted in reference [22], the energy absorbing boundaries perform poorly in transmitting surface waves. The reflections observed in the finite element analysis are attributed to the limited performance of the silent elements.

5. COMPARISON OF TRANSIENT RESPONSES: EFFECT OF WEAK COUPLING

As in the one-dimensional problem, the maximum values of the absolute values of the stresses at the sensor locations are of interest for comparison purposes. In the case of the axisymmetric problem, stress waves travel in the radial direction \( r \), as well as in the axial direction \( z \). Examining the structures of the propagation and attenuation zones of the systems under consideration, it is found that for weakly coupled systems, the value of \( \tau_T \) (or equivalently, \( \tau_L \)) governs the strength of the coupling in both radial and longitudinal directions. Since \( \tau_T \) and \( \tau_L \) (cf. Appendix A) are of the same order, either one can be used as the control parameter. The values of \( v_T \) and \( v_L \) are of order one for the systems under consideration. In the frequency–wavenumber \( (\omega, k) \) plane, the sizes of the propagation zones of the weakly coupled medium (system I) are considerably smaller than those of the moderately coupled medium (system II) as depicted in Figures 4 and 5.

In order to visualize the effects of \( \tau_T \) (or equivalently, \( \tau_L \)) and the structures of the propagation zones on the transient responses, the minimum and maximum values of the stress components \( \sigma_{zz} \) and \( \sigma_{zr} \) are compared for systems I and II. The effects of the structures of the propagation zones on the transient waves are illustrated in Figures 20 and 21, and considerable difference between the transient responses of systems I and II are found. The minimum values of \( \sigma_{zz} \) for system I are over two times larger than those of system II; the maximum values of \( \sigma_{zz} \) for system II are up to six times larger than those of system I. The effect of coupling on the shear stress \( \sigma_{zr} \) is more dominant than that on the maximum longitudinal stress \( \sigma_{zz} \). From these results, the governing role of the size of the propagation zones in the frequency–wavenumber \( (k, \omega) \) plane is clear: the smaller the size of the propagation zones (or the larger the parameter \( \tau_T \) or \( \tau_L \)), the greater the reductions in the maximum and minimum values of the stress components \( \sigma_{zz} \) and \( \sigma_{zr} \). This is in accordance with the results obtained in the one-dimensional case.

Figure 20. Minimum values of \( \sigma_{zz} \) at the four sensor locations at \( r = 0.0, 0.25, 0.5 \) and \( 1.0 \mu \) \( r = 0.0, 6.35, 12.7 \) and 25.4 mm) on the four interfaces of systems I and II. Note that the maximum value of the stress decreases with increasing interface number \( (\sigma_0 = 1.0 \text{ Pa}) \). (a) System I; (b) system II.
When weak coupling between layers exists (τ_T or τ_L are large), the reflectivity between the layers of a bi-periodic set is increased. Hence, the larger the value of τ_T (or τ_L) the weaker the coupling between layers and the lower the value of the reflection coefficient for waves travelling in the positive z-direction between layers A and B. Note that for reflected waves travelling in the negative z-direction, the transmission coefficient is higher compared to waves travelling in the positive z-direction. Consequently, stress waves travelling backwards encounter less resistance against transmission, and stress localization takes place in the top layers of the weakly coupled system (system I). For the reflected waves, the higher the values of τ_T, the stronger the resulting localization is.

Moreover, from Figure 20, it is noted that the effect of weak coupling is more dominant at interfaces close to the circular area where the pressure field is applied. The deeper the waves propagate, the less visible the effect of the coupling becomes. This especially holds for the stress component σ_z, since the applied force primarily excites longitudinal waves. Apparently, in the weakly coupled system, waves are trapped close to the top layers, and the interactions between incident and reflected waves tend to reduce the magnitudes of the resulting waves.

The value of τ_T is 89.9 for system I, and 8.99 for system II. Considering the corresponding one-dimensional pressure (σ_z) wave propagation, the reflection and transmission coefficients for system I in the z-direction are −0.9988 and 0.0651, respectively. For system II, the corresponding reflection and transmission coefficients are −0.4594 and 0.5405, respectively. For both systems, the pressure waves convert into tensile waves when reflected from the interface between layers A and B. The reflected tensile waves can cause fracture in layer A, and debris of layer A can fly in the negative z-direction. This effect is known as the Hopkinson phenomenon [23, 24].

6. CONCLUSIONS

The stress wave propagation in finite bi-periodic layered systems has been investigated. The transfer matrix formulation of the axisymmetric problem is developed using a double integral transform technique. The propagation and attenuation zones of the one- and two-dimensional layered media are studied by employing both analytical and computational methods. The effect of weak coupling between layers in the structure of propagation zones is analyzed in both the transform (frequency–radial wavenumber) and physical (temporal–spatial) domains. It has been shown that the structure of the propagation zones is controlled by a number of physical parameters. The transient...
responses of the finite one- and two-dimensional layered media have been computed with a double integral transformation. For verification purposes, a finite element analysis has been performed.

The emphasis of this work is on the relationship between transformed (frequency–radial wavenumber) and physical (temporal–spatial) domains. The main contribution of the current study is the computational study of the effect of the structure of the propagation zones of the infinite system on the transient behavior of the corresponding finite system. A set of tools to perform the inversion of double integral transforms is developed. The direct global matrix formulation is implemented to avoid the inherent numerical instability problem (exponential dichotomy). While the inverse Fourier transform is performed using the FFT algorithm, an integral technique is adopted for the inverse Hankel transforms. Two configurations termed systems I and II are considered. The layers of the bi-periodic set in system I are weakly coupled. The strength of coupling in the second system, system II, is moderate. The propagation zones of the corresponding infinite systems are computed and compared. The following observations from the transient analysis are obtained: the minimum values of the stress component $s_{zz}$ for system I are up to six times larger than those of system II, and the maximum values of the stress component $s_{rr}$ for system II are up to twice as large as those of system I. These results reveal the important role of the structures of propagation zones on the transient responses of finite periodic systems. The smaller the propagation zones, the greater the reduction in the maximum/minimum values of the stress components. It is also found that the effect of weak coupling diminishes away from the leading layer where the pressure field is applied. Another important finding is that the minimum values of $s_{rr}$ diminish in the radial direction more than those of $s_{zz}$. The maximum values of $s_{zz}$ are maximum at $r = 25.4$ mm, and zero at the epicenter due to the symmetry of the problem.

A transient finite element analysis is also performed to verify the results. Problems associated with transient finite element analyses, such as node-to-node oscillation and silent boundary elements are addressed. Good agreement between results from the double integral technique and finite element method has been obtained especially in the early phase of the stress wave propagation. The limitations of the finite element method on transient analysis are addressed. Another verification was to compare the responses for $k = 0$ with the dynamics of the one-dimensional results obtained in reference [1], where the results obtained from the spectral and modal analyses were found to be almost identical for the one-dimensional wave propagation problem.

To improve the finite element results, the $p$-adaptive method can be employed. Higher order elements can minimize the numerical noise caused by strain discontinuities since in wave propagation problems the strain fields vary smoothly with time. Also, an analysis-based explicit integration method can be executed to study the wavefront of the stress field more accurately.

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REFERENCES


APPENDIX A: ELASTIC AND GEOMETRIC PROPERTIES OF SYSTEMS I AND II

Layer A (‘‘stiff’’ layer; this layer is the same as for both systems I and II.):

\[ E_A = 310\cdot26 \text{ GPa}, \quad \nu_A = 0.25, \quad \rho_A = 3248.81 \text{ kg/m}^3, \]
\[ H_A = 12.7 \text{ mm}, \quad \lambda_A = 124.11 \text{ GPa}, \quad \mu_A = 124.11 \text{ GPa}, \]
\[ c_{T_A} = 6180.63 \text{ m/s}, \quad c_{L_A} = 10705.17 \text{ m/s}, \quad a_A = c_{L_A}/c_{T_A} = 1.73. \]
Layer B ("soft" layer; systems I and II differ in only the values of $E$ and $\rho$):

\[
\begin{align*}
E_\text{B} &= 137\,89 \text{ MPa for system I} = 1378\,95 \text{ MPa for system II,} \\
\nu_\text{B} &= 0.48, \\
\rho_\text{B} &= 1068\,69 \text{ kg/m}^3 \text{ for system I} = 10686\,89 \text{ kg/m}^3 \text{ for system II,} \\
\lambda_\text{B} &= 11\,11 \text{ GPa for system I} = 11\,18 \text{ GPa for system II,} \\
\mu_\text{B} &= 46\,59 \text{ MPa for system I} = 465\,6 \text{ MPa for system II,} \\
\lambda_{\text{c}} = 208\,79 \text{ m/s for systems I and II,} \\
\mu_{\text{c}} = 1064\,61 \text{ m/s for systems I and II,} \\
\zeta_\text{B} &= 5.10 \text{ for systems I and II}
\end{align*}
\]

\[
\begin{align*}
\tau_T &= 89.9, \\
\tau_L &= 30.5, \\
\nu_T &= 0.59, \\
\nu_L &= 0.2 \text{ for system I;}
\end{align*}
\]

\[
\begin{align*}
\tau_T &= 89.9, \\
\tau_L &= 3.05, \\
\nu_T &= 0.59, \\
\nu_L &= 0.2 \text{ for system II.}
\end{align*}
\]

APPENDIX B: COMPOSITE TRANSFER MATRIX $T_{\text{set}}$ FOR A BI-PERIODIC ELEMENT

The entries of the transfer matrix for a generic layer are analytically computed as follows (cf. references [2, 6]):

\[
\begin{align*}
T_{\text{layer}}(1, 1) &= \frac{1}{p^2}(-2k^2 \cosh(g) + (h^2 + k^2) \cosh(h)), \\
T_{\text{layer}}(1, 2) &= \frac{1}{p^2}(-\frac{k}{g} (h^2 + k^2) \sinh(g) + 2hk \sinh(h)), \\
T_{\text{layer}}(1, 3) &= \frac{H}{p^2}(-\frac{k}{\mu} \cosh(g) + \frac{k}{\mu} \cosh(h)), \\
T_{\text{layer}}(1, 4) &= \frac{H}{p^2}(-\frac{k^2 \sinh(g)}{g\mu} + \frac{h}{\mu} \sinh(h)), \\
T_{\text{layer}}(2, 1) &= \frac{1}{p^2}(2gk \sinh(g) - \frac{k}{h} (h^2 + k^2) \sinh(h)), \\
T_{\text{layer}}(2, 2) &= \frac{1}{p^2}((h^2 + k^2) \cosh(g) - 2k^2 \cosh(h)), \\
T_{\text{layer}}(2, 3) &= \frac{H}{p^2}(\frac{g}{\mu} \sinh(g) - \frac{k^2 \sinh(h)}{h\mu}), \\
T_{\text{layer}}(2, 4) &= \frac{H}{p^2}(\frac{k}{\mu} \cosh(g) - \frac{k}{\mu} \cosh(h)), \\
T_{\text{layer}}(3, 1) &= \frac{1}{Hp^2}(2k(h^2 + k^2) \mu \cosh(g) - 2k(h^2 + k^2) \mu \cosh(h)), \\
T_{\text{layer}}(3, 2) &= \frac{1}{Hp^2}(\frac{k}{\mu} (h^2 + k^2) \mu \sinh(g) - 4hk^2 \mu \sinh(h)).
\end{align*}
\]
\[ T_{\text{layer}}(3, 3) = \frac{1}{p^2} ((h^2 + k^2) \cosh(g) + 2k^2 \cosh(h)), \]

\[ T_{\text{layer}}(3, 4) = \frac{1}{p^2} \left( \frac{k}{g} (h^2 + k^2) \sinh(g) - 2hk \sinh(h) \right), \]

\[ T_{\text{layer}}(4, 1) = \frac{1}{Hp^2} \left( -4gk^2 \mu \sinh(g) + \frac{\mu}{h} (h^2 + k^2)^2 \sinh(h) \right), \]

\[ T_{\text{layer}}(4, 2) = \frac{1}{Hp^2} \left( -2k(h^2 + k^2)\mu \cosh(g) + 2k(h^2 + k^2)\mu \cosh(h) \right), \]

\[ T_{\text{layer}}(4, 3) = \frac{1}{p^2} \left( -2gk \sinh(g) + \frac{k}{h} (h^2 + k^2) \sinh(h) \right), \]

\[ T_{\text{layer}}(4, 4) = \frac{1}{p^2} ( -2k^2 \cosh(g) + (h^2 + k^2) \cosh(h)), \]

where \( T_{\text{layer}}(i, j) \) denotes the \( ij \)th entry of the matrix \([T_{\text{layer}}]\).

The two layers, labelled as layer A and B, are made of two different materials, and generally are of different thicknesses. For layer A, the transfer matrix is computed with the substitutions \( H = H_A, a = a_A, \mu = \mu_A, \) and \( g = g_A \) (i.e., the elastic and geometric properties of layer A). The resulting transfer matrix of layer A will be denoted as \([T_A]\).

For the transfer matrix for layer B, the following scaled variables are introduced:

\[ \rho = r/H_A, \quad \xi = z/H_B, \quad \tau = (c_r/H_A)\tau, \]

where the subscripts A and B denote layer A and layer B, respectively. Transforming the wave equations, and the scaled displacement and stress components for layer B, one obtains a transfer matrix similar to \([T_A]\). In order to obtain the transfer matrix of layer B, \([T_B]\), from the expression of \([T_A]\), the following substitutions must be performed:

1. replace \( a_A \) with \( a_B \), i.e., \( g_A \) with \( g_B \), where \( g_B^2 = g_A^2 + p^2/a_A^2 \),
2. replace \( \mu_A \) with \( \mu_B \),
3. multiply the argument of the hyperbolic functions in the entries of the transfer matrix \([T_A]\) by \((H_B/H_A)\).

This procedure provides the transfer matrix for layer B, \([T_B]\). Considering the bi-periodic set, the composite transfer matrix is expressed as \([T_{\text{set}}] = [T_B][T_A]\).