Norm Conditions for Uniform Algebra Isomorphisms

Aaron Luttman*[1], Scott Lambert†[2]

[1] *Bethany Lutheran College,*
Division of Science and Mathematics
700 Luther Drive,
Mankato, MN, 56001, USA

[2] *University of Montana,*
Department of Mathematical Sciences (MMAI01)
Mathematics Building
Missoula, MT, 59812-0864, USA

Abstract: In recent years much work has been done analyzing maps, not assumed to be linear, between uniform algebras that preserve the norm, spectrum, or subsets of the spectra of algebra elements, and it is shown that such maps must be linear and/or multiplicative. Letting A and B be uniform algebras on compact Hausdorff spaces X and Y, respectively, it is shown here that if $\lambda \in \mathbb{C} \setminus \{0\}$ and $T\colon A \to B$ is a surjective map, not assumed to be linear, satisfying

$$\|T(f)T(g) + \lambda\| = \|fg + \lambda\| \quad \forall\ f, g \in A,$$

then T is an $\mathbb{R}$-linear isometry and there exist an idempotent $e \in B$, a function $\kappa \in B$ with $\kappa^2 = 1$, and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}(1-e)$ such that

$$T(f) = \kappa \left(\tilde{T}(f)e + \gamma \overline{\tilde{T}(f)}(1-e) \right)$$

for all $f \in A$, where $\gamma = \frac{\lambda}{|\lambda|}$. Moreover, if T is unital, i.e. $T(1) = 1$, then $T(i) = i$ implies that T is an isometric algebra isomorphism whereas $T(i) = -i$ implies that T is a conjugate-isomorphism.

Keywords: uniform algebras, peripheral spectrum, isometric algebra isomorphism
MSC (2000): 46J10, 46J20, 46H40

* luttman@blc.edu
† scott.lambert@umontana.edu

1 Introduction

A great deal of work has been done on what are generally called "linear preserver" problems in Banach algebras. Such results demonstrate that a given linear map between Banach algebras must preserve some structure - such as the norm, the spectrum, or a subset of the spectrum - or that linear maps that preserve such structure must also be multiplicative (see [11] and the references therein). Recently there has been a move towards addressing these problems without the assumption of linearity on the map, and it is shown that such maps are nonetheless linear and multiplicative or weighted composition operators.

In [9], Kowalski and Słodkowski showed that, if A is a complex Banach algebra (not assumed to be unital or commutative), then any map $T\colon A \to \mathbb{C}$ that satisfies $T(0) = 0$ and $T(f) - T(g) \in \sigma(f - g)$, where $\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda \in A^{-1}\}$ is the spectrum of f, is linear and multiplicative. Later it was shown by Molnár [14] that a surjective map $T\colon C(X) \to C(X)$, for a first-countable compact Hausdorff space X, that satisfies $\sigma(T(f)T(g)) = \sigma(fg)$ is a weighted composition operator, and, in particular, an isometric algebra isomorphism if T is unital (i.e. $T(1) = 1$). Rao and Roy extended these results in [15, 16]. Further work has been done analyzing maps between uniform algebras, Lipschitz algebras, and more general semi-simple commutative Banach algebras that satisfy $\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$, where $\sigma_\pi(f) = \{\lambda \in \sigma(f) : |\lambda| = \|f\|\}$ is the peripheral spectrum of f (see [2, 4, 5, 8, 12]).

Most recently, work has been done analyzing other maps that polynomially preserve the spectrum or peripheral spectrum in the sense $\sigma(p(T(f), T(g))) = \sigma(p(f, g))$ or $\sigma_\pi(p(T(f), T(g))) = \sigma_\pi(p(f, g))$ for some polynomial $p(z, w)$. Rao, Tonev, and Toneva in [17] showed that if a surjective map $T\colon A \to B$ satisfies $\sigma_\pi(T(f) + T(g)) = \sigma_\pi(f + g)$ for all $f, g \in A$ and also satisfies a criterion on the maximum modulus of sums of elements, then T is an isometric algebra isomorphism. In [4], Hatori, Miura, and Takagi have more generally classified certain polynomials for which $\sigma(p(T(f), T(g)) = \sigma(p(f, g))$ implies T must be an isometric algebra isomorphism. Generalizing further to norm conditions, Lambert, Luttman, and Tonev [10] showed that a surjective, homogeneous map satisfying $\|T(f)T(g)+1\| = \|fg+1\|$ is an isometric algebra isomorphism, and Honma [6] has shown that a surjective, unital map $T\colon C(X) \to C(Y)$ that satisfies $\|T(f)\overline{T(g)} - 1\| = \|f\overline{g} - 1\|$ must be an isometric algebra isomorphism.

More generally, Hatori, Miura, and Takagi have characterized maps $T\colon A \to B$ between semisimple commutative Banach algebras that satisfy $r(fg - 1) = r(T(f)T(g) - 1)$ for all $f, g \in A$, where $r(f)$ is the spectral radius of f. Denoting by δA the Choquet boundary of A, we give the uniform algebra formulation of their theorem [3, Corollary 7.5].

Theorem. *Let A and B be uniform algebras on compact Hausdorff spaces X, and Y, respectively. If $\lambda \in \mathbb{C}\backslash\{0\}$ and $T\colon A \to B$ is a surjective map satisfying $\|T(f)T(g) - \lambda\| = \|fg - \lambda\|$ for all $f, g \in A$, then there exists a homeomorphism $\varphi\colon \delta B \to \delta A$, a continuous* *function* $\eta\colon \delta B \to \{\pm 1\}$*, and a clopen set* $K \subset \delta B$ *such that*

$$T(f)(y) = \eta(y) \times \begin{cases} f(\varphi(y)), & y \in K \\ \frac{\lambda}{|\lambda|}\overline{f(\varphi(y))}, & y \in \delta B \setminus K \end{cases}$$

holds for every $f \in A$ *and every* $y \in \delta B$*.*

Rather than a pointwise characterization of the mapping T, it is also possible to characterize T as an operator (not necessarily linear). We thus give in this work an alternate characterization and a simplified proof of the above result and prove the following theorem:

Main Theorem. *Let* A *and* B *be uniform algebras on compact Hausdorff spaces* X *and* Y*, respectively, and let* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $\gamma = \frac{\lambda}{|\lambda|}$*. If* $T\colon A \to B$ *is a surjective map, not assumed to be linear, and*

$$\|T(f)T(g) + \lambda\| = \|fg + \lambda\| \quad \forall\ f, g \in A, \tag{1}$$

then there exist an idempotent $e \in B$*, a function* $\kappa \in B$ *with* $\kappa^2 = 1$*, and an isometric algebra isomorphism* $\tilde{T}\colon A \to Be \oplus \overline{B}(1-e)$ *such that*

$$T(f) = \kappa\left(\tilde{T}(f)e + \gamma\overline{\tilde{T}(f)}(1-e)\right)$$

for all $f \in A$*.*

We begin by providing definitions and notations required for the above results. The set of *peaking functions* of a uniform algebra A is $\mathcal{F}(A) = \{h \in A : \sigma_\pi(h) = \{1\}\}$. If $h \in A$ is a peaking function, then the set of points for which h takes the value 1 is called the *peak set* of h and is denoted by $P(h) = \{x \in X : h(x) = 1\}$. The set of peaking functions that peak at a point $x_0 \in X$ are denoted by $\mathcal{F}_{x_0}(A) = \{h \in \mathcal{F}(A) : x_0 \in P(h)\}$. A point $x_0 \in X$ that is the intersection of peak sets is called a p-point, and the set of p-points is well-known to form a boundary for A, i.e. a set on which all functions $f \in A$ attain their maximum modulus. A simple proof of this fact is given in [10, Lemma 5]. The set of p-points is the Choquet boundary of A, denoted by δA, which is dense in the Shilov boundary, the smallest closed boundary of A. If X is metrizable, the Choquet boundary is the minimal boundary of A under inclusion.

An essential tool used in results of this type is the following result due to Bishop:

Lemma 1.1. [1, Theorem 2.4.1] *For every peak set* $E \subset X$ *and every* $f \in A$ *with* $f|_E \neq 0$*, there exists a peaking function* $h \in \mathcal{F}(A)$ *such that* $P(h) = E$ *and* fh *attains its maximum modulus exclusively on* E*.*

It is well known that the exponent of a uniform algebra, $e^A = \{e^f : f \in A\}$, consists of invertible elements, and we now show that the peaking function h given by Bishop's Lemma can actually be chosen to be an element of e^A.

Corollary 1.2. *If $E \subset X$ is a peak set, and $f \in A$ is such that $f|_E \neq 0$, then there exists $h \in \mathcal{F}(A) \cap e^A$ such that fh attains its maximum modulus exclusively on E. In particular, h is invertible.*

Proof. We will use the following inequality, easily verified with Rolle's Theorem.

$$e^{n(x-1)} < x, \quad \forall\, x \in \left[2^{-1}, 1\right), \quad \forall\, n \geq 2 \tag{2}$$

Let E be a peak set and $f \in A$ such that $f|_E \neq 0$. By Lemma 1.1 there exists $h \in \mathcal{F}(A)$ such that $P(h) = E$ and fh takes its maximum modulus only on E. Choose $n \geq 2$ such that $\left(e^{-\frac{1}{2}}\right)^n < \frac{\|fh\|}{\|f\|}$ and define $k = e^{n(h-1)}$. Firstly, for all $x \in X$, $|k(x)| = e^{n(\operatorname{Re} h(x)-1)} \leq e^{\operatorname{Re} h(x)-1} \leq 1$ since $\operatorname{Re} h(x) - 1 \leq 0$. Also

$$1 = |k(x)| \iff n(\operatorname{Re} h(x) - 1) = 0 \iff \operatorname{Re} h(x) = 1 \iff h(x) = 1,$$

and thus $1 = |k(x)|$ implies $k(x) = 1$. Therefore $k \in \mathcal{F}(A)$ and $P(k) = P(h) = E$. Also for any $x \in X$ with $\|fh\| = |f(x)h(x)|$, we have $x \in P(h) = P(k)$, which implies $\|fh\| = |f(x)h(x)| = |f(x)k(x)| \leq \|fk\|$.

Finally we show that fk attains its maximum modulus exclusively on E. Let $x \in X$ such that $x \notin P(k)$. Then $x \notin P(h)$ and $-1 \leq \operatorname{Re} h(x) < 1$.

Case 1: $\frac{1}{2} \leq \operatorname{Re} h(x) < 1$. Then $|k(x)| = e^{n(\operatorname{Re} h(x)-1)} < \operatorname{Re} h(x) \leq |h(x)|$, by (2), so $|f(x)k(x)| < |f(x)h(x)| \leq \|fh\| \leq \|fk\|$.

Case 2: $\operatorname{Re} h(x) < \frac{1}{2}$. Then $\operatorname{Re} h(x) - 1 < -\frac{1}{2}$ and $|k(x)| = \left(e^{\operatorname{Re} h(x)-1}\right)^n < \left(e^{-\frac{1}{2}}\right)^n < \frac{\|fh\|}{\|f\|}$. Thus $|f(x)k(x)| < \|f\| \frac{\|fh\|}{\|f\|} = \|fh\| \leq \|fk\|$.

Since $|f(x)k(x)| < \|fk\|$ for all $x \notin P(k) = E$, fk attains its maximum modulus exclusively on E.

2 Reduced Norm Condition

In order to prove the main theorem, we first analyze surjective maps that satisfy

$$\|T(f)T(g) + 1\| = \|fg + 1\|, \tag{3}$$

and this will be used to prove the general case.

Lemma 2.1 (Identification Lemma)**.** *Let A be a uniform algebra on a compact Hausdorff space X and $f, g \in A$. If $\|fk + 1\| = \|gk + 1\|$ for all $k \in A$, then $f = g$.*

Proof. Suppose that $\|fk + 1\| = \|gk + 1\|$ for all $k \in A$ but that $f \neq g$. Then there exists $x_0 \in \delta A$ such that $f(x_0) \neq g(x_0)$.

We prove two cases. First suppose that $|f(x_0)| < |g(x_0)|$. Then there exists a neighborhood V of x_0 such that $|f| < |g|$ on V. By Corollary 1.2 there exists $h \in \mathcal{F}_{x_0}(A)$ such that $P(h) \subset V$ and gh and fh attain their maximum moduli only on $P(h)$, and note that, since $|f| < |g|$ on V, $\|fh\| < \|gh\|$. Let $x' \in P(h)$ be such that $|g(x')| = \|gh\|$, and suppose that $g(x') = re^{i\theta}$. Then

$$\|gh\| + 1 = \left\|g\left(e^{-i\theta}h\right) + 1\right\| = \left\|f\left(e^{-i\theta}h\right) + 1\right\| \leq \|fh\| + 1 < \|gh\| + 1,$$

a contradiction.

In the second case, $|f| = |g|$ on δA and $f \neq g$. Since $f(x_0) = re^{i\theta}$, we have that $g(x_0) = re^{i\gamma}$ for some $\gamma \neq \theta$. Since f and g are continuous, there exists an open neighborhood V of x_0, such that $f(V) \cap g(V) = \emptyset$. By Corollary 1.2 there exists $h \in \mathcal{F}_{x_0}(A)$ such that $P(h) \subset V$ and fh attains its maximum modulus only on $P(h)$. Since $|f| = |g|$, gh also attains its maximum modulus only on $P(h)$. Let $x' \in P(h)$ be such that $|f(x')| = \|fh\|$, and suppose that $f(x') = ce^{i\mu}$. Then, since $g(x) \neq ce^{i\mu}$ for any $x \in V$,

$$\|gh\| + 1 = \|fh\| + 1 = \left\|f\left(e^{-i\mu}h\right) + 1\right\| = \left\|g\left(e^{-i\mu}h\right) + 1\right\| \neq \|gh\| + 1,$$

a contradiction. Thus $f = g$.

Note that the proof requires the condition be met only for all k of the form $k = e^{i\theta}h$ with $\theta \in [0, 2\pi)$ and $h \in \mathcal{F}(A) \cap e^A \subset A^{-1}$.

In the remainder of this section we assume that $T\colon A \to B$ is a surjective map satisfying (3).

Lemma 2.2. *The map T satisfies*

(a) T is injective,

(b) $f \in A^{-1}$ if and only if $T(f) \in B^{-1}$, and

(c) $\sigma(T(i)) \subset \{\pm i\}$ and $\sigma(T(-i)) \subset \{\pm i\}$.

Proof. Suppose $T(f) = T(g)$ for $f, g \in A$. Then

$$\|fk + 1\| = \|T(f)T(k) + 1\| = \|T(g)T(k) + 1\| = \|gk + 1\|$$

for all $k \in A$. By the Identification Lemma 2.1, $f = g$, which proves (a). Since T is bijective, there exists a well-defined, bijective map $T^{-1}\colon B \to A$ such that $\|T^{-1}(f)T^{-1}(g) + 1\| = \|fg + 1\|$ for all $f, g \in B$.

Let $f \in A^{-1}$. Then $0 = \|(f)\left(-f^{-1}\right) + 1\| = \|T(f)T\left(-f^{-1}\right) + 1\|$. Thus $T(f)T\left(-f^{-1}\right) = -1$, which implies that $T(f)$ is invertible and

$$T(f)^{-1} = -T\left(-f^{-1}\right). \tag{4}$$

A similar argument applied to T^{-1} completes the proof of (b).

Lastly, notice that

$$0 = \|-1 + 1\| = \|i^2 + 1\| = \|T(i)^2 + 1\|,$$

which shows that $T(i)^2 = -1$. Thus $\sigma(T(i)) \subset \{\pm i\}$. A similar argument holds for $T(-i)$, proving (c).

A mapping T is called *norm-multiplicative* if it satisfies $\|fg\| = \|T(f)T(g)\|$ for all $f, g \in A$. Following an argument similar to that of Honma [6, Lemma 3.3], we next show that T satisfies the norm-multiplicative property when at least one of factors is invertible.

Lemma 2.3. *For all $f \in A$ and $g \in A^{-1}$, $\|T(f)T(g)\| = \|fg\|$ and $\|T(f)\| = \|f\|$.*

Proof. Let $g \in A^{-1}$ and $K_n = T(-ng)T(g)^{-1}$. Then

$$\|K_n\| - 1 \leq \|K_n - 1\| = \|T(-ng)T(g)^{-1} - 1\| = \left\|T(-ng)\left(-T(g)^{-1}\right) + 1\right\|.$$

By (4), $-T(g)^{-1} = T\left(-g^{-1}\right)$, so

$$\left\|T(-ng)\left(-T(g)^{-1}\right) + 1\right\| = \left\|T(-ng)T\left(-g^{-1}\right) + 1\right\| = \left\|-ng\left(-g^{-1}\right) + 1\right\| = n + 1.$$

Thus $\|K_n\| \leq n + 2$. Let $f \in A$, then

$$\begin{aligned} n\|fg\| - 1 &\leq \|fng - 1\| = \|f(-ng) + 1\| = \|T(f)T(-ng) + 1\| \leq \|T(f)T(-ng)\| + 1 \\ &= \|T(f)K_nT(g)\| + 1 \leq \|K_n\|\|T(f)T(g)\| + 1 \leq (n+2)\|T(f)T(g)\| + 1. \end{aligned}$$

Simplifying yields

$$\|fg\| \leq \frac{n+2}{n}\|T(f)T(g)\| + \frac{2}{n},$$

which shows that $\|fg\| \leq \|T(f)T(g)\|$ by letting $n \to \infty$. A similar argument with T^{-1} shows that $\|T(f)T(g)\| \leq \|fg\|$, yielding $\|fg\| = \|T(f)T(g)\|$. The second assertion follows immediately noting that $|T(i)| = 1$.

Lemma 2.4. *For all $f \in A$ and $g \in A^{-1}$, $1 \in \sigma_\pi(T(f)T(g))$ if and only if $1 \in \sigma_\pi(fg)$.*

Proof. Suppose that $1 \in \sigma_\pi\left(T(f)T(g)\right)$. Then

$$\|T(f)T(g)\| + 1 = \|T(f)T(g) + 1\| = \|fg + 1\| \leq \|fg\| + 1 = \|T(f)T(g)\| + 1,$$

where the last equality holds by Lemma 2.3. Thus $\|fg\| + 1 = \|fg + 1\|$, which, along with $\|fg\| = 1$, implies $1 \in \sigma_\pi(fg)$. The other direction follows similarly.

Lemma 2.5. *For all $\alpha, \beta \in \mathbb{C}$ and all $y \in Y$,*
(a) $|T(\alpha)(y)| = |\alpha|$,
(b) $\operatorname{Re} T(\alpha)(y)T(\beta)(y) \leq \operatorname{Re} \alpha\beta$,
(c) $T(-i) = -T(i)$, *and*
(d) $T(1)^2 = 1$.

Proof. Take $\alpha \in \mathbb{C}$ and $y \in Y$. If $\alpha = 0$, by Lemma 2.3, $T(0) = 0$. Otherwise $\alpha \in A^{-1}$ and $|\alpha^{-1}| = |\alpha|^{-1}$ so

$$|T(\alpha)^{-1}(y)| \leq \|T(\alpha)^{-1}\| = \| - T(-\alpha^{-1})\| = \|\alpha^{-1}\| = |\alpha|^{-1}.$$

Thus $|\alpha| \le |T(\alpha)(y)| \le \|T(\alpha)\| = |\alpha|$, proving (a).

Now for all $\alpha, \beta \in \mathbb{C}$, $|T(\alpha)(y)T(\beta)(y)+1| \le \|T(\alpha)T(\beta)+1\| = |\alpha\beta+1|$. Squaring both sides gives $(\operatorname{Re} T(\alpha)(y)T(\beta)(y) + 1)^2 + (\operatorname{Im} T(\alpha)(y)T(\beta)(y))^2 \le (\operatorname{Re} \alpha\beta + 1)^2 + (\operatorname{Im} \alpha\beta)^2$ which yields $|T(\alpha)(y)|^2|T(\beta)(y)|^2 + 2\operatorname{Re} T(\alpha)(y)T(\beta)(y) \le |\alpha|^2|\beta|^2 + 2\operatorname{Re}\alpha\beta$. Thus (b) follows, since $|T(\alpha)(y)| = |\alpha|$ and $|T(\beta)(y)| = |\beta|$.

From Lemma 2.2 (c), $\sigma(T(i)) \subset \{\pm i\}$, and $\sigma(T(-i)) \subset \{\pm i\}$. By examining the cases we get $T(-i)(y) = T(i)(y)$ or $T(-i)(y) = -T(i)(y)$ for all $y \in Y$. In fact, if we define $E = \frac{1-T(i)T(-i)}{2}$ then

$$E(y) = \begin{cases} 1 & \text{if } \; T(-i)(y) = T(i)(y) \\ 0 & \text{if } \; T(-i)(y) = -T(i)(y). \end{cases}$$

The lemma follows if $E = 0$. Suppose $E \ne 0$ then $\|E\| = 1$. Clearly $T(i)E = T(-i)E$. Let $e = T^{-1}(T(i)E)$ and $\lambda \in \sigma_\pi(e)$. Since $1 = \|E\| = \|e\|$, $|\lambda| = 1$ and $1 \in \sigma_\pi(\overline{\lambda}e)$. Applying Lemma 2.4 gives $1 \in \sigma_\pi(T(\overline{\lambda})T(e)) = \sigma_\pi(T(\overline{\lambda})T(i)E)$, i.e. $1 = T(\overline{\lambda})(y')T(i)(y')$ for some $y' \in P(E)$. By (b), $1 = \operatorname{Re} T(\overline{\lambda})(y')T(i)(y') \le \operatorname{Re}\overline{\lambda}i = \operatorname{Im}\lambda$. Thus $\sigma_\pi(e) = \{\lambda\} = \{i\}$. This gives $1 \in \sigma_\pi(-ie)$, so by Lemma 2.4 $1 \in \sigma_\pi(T(-i)T(e)) = \sigma_\pi(T(-i)T(-i)E) = \sigma_\pi(-E) = \{-1\}$, which is a contradiction, proving (c).

By (b), $\operatorname{Re} T(1)(y)T(i)(y) \le 0$ and $-\operatorname{Re} T(1)(y)T(i)(y) = \operatorname{Re} T(1)(y)T(-i)(y) \le 0$ and so $\operatorname{Re} T(1)(y)T(i)(y) = 0$ for all $y \in Y$. Since $T(i)(y) = i$ or $-i$, $\operatorname{Im} T(1)(y) = 0$, so (d) follows from (a).

Lemma 2.6. *If $T(1) = 1$ and $T(i) = i$, then $T(\alpha) = \alpha$ for all $\alpha \in \mathbb{C}$.*

Proof. By (4), $T(-1) = -1$, and by Lemma 2.5 (c), $T(-i) = -i$. Applying Lemma 2.5 (b) successively with $\beta = 1, -1, i, -i$ yields

$$\begin{array}{rclcrcl} \operatorname{Re} T(\alpha) & \le & \operatorname{Re}\alpha & \qquad & -\operatorname{Re} T(\alpha) & \le & -\operatorname{Re}\alpha \\ \operatorname{Re} iT(\alpha) & \le & \operatorname{Re} i\alpha & & -\operatorname{Re} iT(\alpha) & \le & -\operatorname{Re} i\alpha, \end{array}$$

which imply $\operatorname{Re} T(\alpha) = \operatorname{Re}\alpha$ and $\operatorname{Im} T(\alpha) = \operatorname{Im}\alpha$.

Lemma 2.7. *Under the hypotheses of Lemma 2.6, $\sigma_\pi(f) = \sigma_\pi(T(f))$ for all $f \in A$. Furthermore, $T(\mathcal{F}(A)) = \mathcal{F}(B)$.*

Proof. Let $\alpha \in \mathbb{C} \setminus \{0\}$. By Lemmas 2.4 and 2.6, $1 \in \sigma_\pi(\alpha^{-1}f)$ if and only if $1 \in \sigma_\pi(T(\alpha^{-1})T(f)) = \sigma_\pi(\alpha^{-1}T(f))$, which implies $\alpha \in \sigma_\pi(f)$ if and only if $\alpha \in \sigma_\pi(T(f))$. Since this clearly also holds for $\alpha = 0$, $\sigma_\pi(f) = \sigma_\pi(T(f))$ for any $f \in A$. That $T(\mathcal{F}(A)) = \mathcal{F}(B)$ follows immediately.

By [10, Theorem 1], if T is surjective, norm multiplicative, and $T(\mathcal{F}(A)) = \mathcal{F}(B)$, then there exists a homeomorphism $\varphi\colon \delta A \to \delta B$, given by

$$\varphi(x) = \bigcap_{h \in \mathcal{F}_x(A)} P(T(h)), \tag{5}$$

satisfying

$$|T(f)(\varphi(x))| = |f(x)| \qquad \forall\, x \in \delta A. \tag{6}$$

Lemma 2.7 gives $T(\mathcal{F}(A)) = \mathcal{F}(B)$. It was only shown in Lemma 2.3 that T is norm-multiplicative when one of the factors is an invertible function, but this is, in fact, sufficient for the proof in [10] to be valid. These results lead to the following lemma, which is a special case of [3, Corollary 7.5], provided here with an alternate proof.

Lemma 2.8. *Let $A \subset C(X)$ and $B \subset C(Y)$ be uniform algebras. If $T\colon A \to B$ is a surjective map, not assumed to be linear, that satisfies $T(1) = 1$ and $T(i) = i$, and (3), then T is an isometric algebra isomorphism.*

Proof. Since T satisfies the hypotheses of [10, Theorem 1], the homeomorphism φ in (5) exists and satisfies (6). We show that T is an isometric algebra isomorphism by proving

$$T(f)(\varphi(x)) = f(x) \tag{7}$$

for all $x \in \delta A$ and $f \in A$.

Let $x \in \delta A$, $f \in A$, and suppose that $T(f)(\varphi(x)) = 0$. Then $f(x) = 0$ by (6), and, since the converse also holds, $T(f)(\varphi(x)) = 0$ if and only if $f(x) = 0$.

Now suppose that $f(x) \neq 0$, and let V be an open neighborhood of x. Then, by Corollary 1.2 there exists an invertible $h \in \mathcal{F}_x(A)$ such that fh attains its maximum modulus only on $P(h) \subset V$, and therefore also $x_1 \in P(h)$ such that $f(x_1) \in \sigma_\pi(fh)$. Letting $\alpha = f(x_1)$ yields $1 \in \sigma_\pi(f\alpha^{-1}h)$, so, by Lemma 2.4, $1 \in \sigma_\pi(T(f)T(\alpha^{-1}h))$. Thus there exists $x_2 \in \delta A$ such that

$$1 = T(f)(\varphi(x_2))T(\alpha^{-1}h)(\varphi(x_2)) \tag{8}$$

and by (6), $1 = |f(x_2)\alpha^{-1}h(x_2)|$, so $\|fh\| = |\alpha| = |f(x_2)h(x_2)|$. Since fh attains its maximum modulus only on $P(h)$, $x_2 \in P(h)$. Therefore

$$|T(\alpha^{-1}h)(\varphi(x_2))| = |\alpha^{-1}h(x_2)| = |\alpha^{-1}| = \|\alpha^{-1}h\| = \|T(\alpha^{-1}h)\|,$$

which implies $T(\alpha^{-1}h)(\varphi(x_2)) \in \sigma_\pi(T(\alpha^{-1}h)) = \sigma_\pi(\alpha^{-1}h) = \{\alpha^{-1}\}$, using Lemma 2.7. Thus (8) becomes $\alpha = T(f)(\varphi(x_2))$, i.e. $f(x_1) = T(f)(\varphi(x_2))$. So for every neighborhood V of x there exist $x_1, x_2 \in V$ such that $f(x_1) = T(f)(\varphi(x_2))$. By the continuity of $f, T(f)$, and $\varphi, T(f)(\varphi(x)) = f(x)$.

We return to analyzing T without assuming it preserves 1 and i. This is done by constructing a transformation of T that preserves both 1 and i, which will prove the Main Theorem in case where $\lambda = 1$.

Theorem 2.9. *Let $T\colon A \to B$ be a surjective map that satisfies $\|T(f)T(g) + 1\| = \|fg + 1\|$ for all $f, g \in A$. Then there exist an idempotent $e \in B$ and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that*

$$T(f) = T(1)\left(\tilde{T}(f)e + \overline{\tilde{T}(f)}e'\right) \tag{9}$$

for all $f \in A$, where $e' = 1 - e$.

Proof. Note that e' is also idempotent and $ee' = e(1-e) = e - e^2 = 0$. This property allows B to be written as the internal direct sum (as rings) of the ideals Be and Be'. Also $Be \oplus \overline{B}e' \subset C(Y)$ is clearly a uniform algebra on Y.

Let $e = \frac{T(1)T(i)+i}{2i}$, then a direct computation, using Lemmas 2.5 (d) and 2.2 (c), verifies that e is an idempotent. Define $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ by

$$\tilde{T}(f) = T(1)\left(T(f)e + \overline{T(f)}e'\right)$$

for all $f \in A$.

To see that $\tilde{T}$ is surjective, take $k \in Be \oplus \overline{B}e'$. There exists $k_1, k_2 \in B$ such that $k_1 e + \overline{k_2}e' = k$. Since T is surjective, there exists $h \in A$ such that $h = T^{-1}\left(T(1)\left(k_1 e + k_2 e'\right)\right)$. Then

$$\tilde{T}(h) = T(1)\left(T(h)e + \overline{T(h)}e'\right) = T(1)^2\left(k_1 e^2 + k_2 e' e + \overline{k_1 ee' + k_2 e'^2}\right) = k_1 e + \overline{k_2}e'.$$

Also, since $\overline{T(1)} = T(1)$ and $\overline{T(i)} = -T(i)$ by Lemmas 2.5 (d) and 2.2 (c), $\tilde{T}(1) = T(1)\big(T(1)e + \overline{T(1)}e'\big) = T(1)^2(e + e') = 1$ and

$$\begin{aligned}\tilde{T}(i) &= T(1)\big(T(i)e + \overline{T(i)}e'\big) = T(1)\big(T(i)e - T(i)e'\big) = T(1)T(i)\big(2e - 1\big)\\ &= T(1)T(i)(-i)T(1)T(i) = -iT(1)^2T(i)^2 = -i(1)(-1) = i.\end{aligned}$$

Moreover, $\tilde{T}$ satisfies the norm condition (3). To see this, note firstly that T satisfies (3), but also $|\tilde{T}(f)(y)\tilde{T}(g)(y)+1| = |T(f)(y)T(g)(y)+1|$ or $|\overline{T(f)(y)T(g)(y)}+1|$ depending on whether $e(y) = 1$ or $e(y) = 0$. Since $|T(f)(y)T(g)(y) + 1| = |\overline{T(f)(y)T(g)(y)} + 1|$, $\|fg + 1\| = \|\tilde{T}(f)\tilde{T}(g) + 1\|$ for all $f, g \in A$. Thus, by Lemma 2.8, $\tilde{T}$ is an isometric algebra isomorphism onto $Be \oplus \overline{B}e'$. To finish the proof note that $\tilde{T}(f)e = T(1)T(f)e$ and $\tilde{T}(f)e' = T(1)\overline{T(f)}e'$. Thus

$$\begin{aligned}T(1)\left(\tilde{T}(f)e + \overline{\tilde{T}(f)}e'\right) &= T(1)\Big(T(1)T(f)e + T(1)T(f)e'\Big)\\ &= T(1)^2T(f)(e + e') = T(f)\end{aligned}$$

which justifies (9) and completes the proof.

3 Main Theorem

We will now prove the Main Theorem using the reduced norm condition result of the previous section.

Theorem 3.1 (Main Theorem)**.** *Let A and B be uniform algebras on compact Hausdorff spaces X and Y, respectively, and let $\lambda \in \mathbb{C} \setminus \{0\}$ and $\gamma = \frac{\lambda}{|\lambda|}$. If $T\colon A \to B$ is a surjective map satisfying $\|T(f)T(g) + \lambda\| = \|fg + \lambda\|$ for all $f, g \in A$, then there exist an* *idempotent $e \in B$, a function $\kappa \in B$ with $\kappa^2 = 1$, and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that*

$$T(f) = \kappa\left(\tilde{T}(f)e + \gamma\overline{\tilde{T}(f)}e'\right)$$

for all $f \in A$, where $e' = 1 - e$.

Proof. Choose α such that $\alpha^2 = \lambda$, and define $T'(f) = \alpha^{-1}T(\alpha f)$. Since α is invertible, T' is surjective, and $\|T'(f)T'(g)+1\| = \|\alpha^{-2}T(\alpha f)T(\alpha g)+1\| = \frac{1}{|\lambda|}\|T(\alpha f)T(\alpha g)+\lambda\| = \frac{1}{|\lambda|}\|\alpha^2 fg + \lambda\| = \|fg+1\|$. By Theorem 2.9 there exist an idempotent $e \in B$ and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that $T'(f) = \kappa\left(\tilde{T}(f)e + \overline{\tilde{T}(f)}e'\right)$, where $\kappa = T'(1)$. Thus

$$\begin{aligned} T(f) = \alpha T'(\alpha^{-1}f) &= \alpha\kappa\left(\tilde{T}(\alpha^{-1}f)e + \overline{\tilde{T}(\alpha^{-1}f)}e'\right) \\ &= \kappa\left(\tilde{T}(f)e + \alpha\overline{\alpha}^{-1}\overline{\tilde{T}(f)}e'\right) = \kappa\left(\tilde{T}(f)e + \gamma\overline{\tilde{T}(f)}e'\right) \end{aligned}$$

since $\frac{\alpha}{\overline{\alpha}} = \frac{\alpha^2}{\alpha\overline{\alpha}} = \frac{\alpha^2}{|\alpha^2|} = \gamma$.

From the formula given for T, it is clear that T is an $\mathbb{R}$-linear isometry. Also using this formula, we may give sufficient conditions for T to be an isomorphism or the conjugate of an isomorphism.

Corollary 3.2. *Let A and B be uniform algebras, $\lambda \in \mathbb{C}\backslash\{0\}$, and $T\colon A \to B$ a surjective map such that*

$$\|T(f)T(g) + \lambda\| = \|fg + \lambda\|$$

for all $f, g \in A$. If
(1) $T(1) = 1$ and $T(i) = i$, then T is an isometric algebra isomorphism.
(2) $T(1) = 1$ and $T(i) = -i$, then T is a conjugate-isomorphism.

Proof. Theorem 3.1 implies $T(1) - iT(i) = \kappa(e + \gamma e') - i\kappa(ie - \gamma ie') = 2\kappa e$. In the first case, $2 = T(1) - iT(i) = 2\kappa e$ implies e is invertible and thus $e = 1$, since it is idempotent. Then $e' = 0, \kappa = 1$ and $T(f) = \tilde{T}(f)$.

In the second case $0 = T(1) - iT(i) = 2\kappa e$, which implies $e = 0$, since κ is invertible. Thus $e' = 1$ and $T(f) = \kappa\gamma\overline{\tilde{T}(f)}$, but $T(1) = 1$ additionally requires that $\kappa\gamma = 1$ so $T(f) = \overline{\tilde{T}(f)}$. Since $\sigma(\kappa) \subset \{\pm 1\}$, $\kappa\gamma = 1$ requires that either $\kappa = 1 = \gamma$ or $\kappa = -1 = \gamma$.

Acknowledgements

The authors would like to thank O. Hatori for suggesting the problem. We would also like to thank T. Tonev for useful discussions and helpful comments on the results as well as the anonymous referees for their suggestions to improve the manuscript.

References

[1] A. Browder, *Introduction to Function Algebras*, W.A. Benjamin Inc. (1969).

[2] O. Hatori, T. Miura and H. Takagi: "Characterization of Isometric Isomorphisms Between Uniform Algebras via Non-linear Range Preserving Properties," *Proc. Amer. Math. Soc.*, Vol. 134 (2006), pp. 2923-2930.

[3] O. Hatori, T. Miura, and H. Takagi: "Multiplicatively Spectrum-Preserving and Norm-Preserving Maps Between Invertible Groups of Commutative Banach Algebras," (2006), preprint.

[4] O. Hatori, T. Miura and H. Takagi: "Polynomially Spectrum-preserving Maps Between Commutative Banach Algebras," (2006), preprint.

[5] O. Hatori, T. Miura and H. Takagi: "Unital and Multiplicatively Spectrum-preserving Surjections Between Semi-simple Commutative Banach Algebras are Linear and Multiplicative," *J. Math. Anal. Appl.*, Vol. 326 (2007), pp. 281-296.

[6] D. Honma: "Norm-Preserving Surjections on Algbras of Continuous Functions," *Rocky Mountain. J. Math.* (2007), to appear.

[7] D. Honma: "Surjections on the Algebras of Continuous Functions which Preserve Peripheral Spectrum," *Contemp. Math.*, Vol. 435 (2007), pp. 199-205.

[8] A. Jiménez-Vargas, A. Luttman, and M. Villegas-Vallecillos: "Weakly Peripherally Multiplicative Surjections of Pointed Lipschitz Algebras," (2008), preprint.

[9] S. Kowalski and Z. Słodkowski: "A Characterization of Maximal Ideals in Commutative Banach Algebras," *Studia Math*, Vol. 67 (1980), pp. 215-223.

[10] S. Lambert, A. Luttman and T. Tonev: "Weakly Peripherally-Multiplicative Operators Between Uniform Algebras," *Contemp. Math.*, Vol. 435 (2007), pp. 265-281.

[11] C-K. Li and N-K. Tsing: "Linear Preserver Problems: A Brief Introduction and Some Special Techniques," *Linear Alegbra Appl.*, Vol. 162-164 (1992), pp. 217-235.

[12] A. Luttman and T. Tonev: "Uniform Algebra Isomorphisms and Peripheral Multiplicativity," *Proc. Amer. Math. Soc.*, Vol. 135 (2007), no. 11, pp. 3589-3598.

[13] S. Mazur and S. Ulam, "Sur les transformations isométriques d'espaces vectoriels normés," *C. R. Acad. Sci.*, Vol. 194 (1932), pp. 946-948.

[14] L. Molnár: "Some Characterizations of the Automorphisms of $B(H)$ and $C(X)$," *Proc. Amer. Math. Soc.*, Vol. 130 (2001), pp. 111-120.

[15] N. V. Rao and A. K. Roy: "Multiplicatively Spectrum Preserving Maps of Function Algebras," *Proc. Amer. Math. Soc.*, Vol. 133 (2005), no. 4, pp. 1135-1142.

[16] N. V. Rao and A. K. Roy: "Multiplicatively Spectrum-preserving Maps of Function Algebras II," *Proc. Edinburgh Math. Soc.*, Vol. 48 (2005), pp. 219-229.

[17] N. V. Rao, T. Tonev and E. Toneva: "Uniform algebra isomorphisms and peripheral spectra," *Contemp. Math.*, Vol. 427 (2007), pp. 401-416.