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Lecture 15: The Interpolation Method

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In this lecture, we introduce the *Cutting Planes* proof system and prove an exponential lower bound using the *interpolation method*. Like Resolution, Cutting Planes is a refutation system. Lines in a Cutting Planes proof are linear inequalities of the form $a_1x_1 + \cdots + a_nx_n \geq A$. New lines can be inferred from previous ones by addition and by multiplication and division by positive integers. The goal is to derive the inequality $0 \geq 1$. This establishes that the initial set of inequalities is unsatisfiable. We will show that Cutting Planes are more powerful than Resolution and no more powerful than the Sequent Calculus and Frege systems.

The interpolation method can be seen as an extension of the decision tree method we used in the previous lecture to prove a lower bound for Tree Resolution. There, we converted a refutation of an unsatisfiable set of clauses into a decision tree of the same size for the following problem: given as assignment to the variables, output a clause that is falsified by that assignment. We then showed that the decision tree has to be large, which implies that the refutation is large.

In this lecture, we convert a Cutting Planes refutation of a set of inequalities of a certain form into a circuit of roughly the same size that computes a related function. By choosing a particular set of inequalities, we will ensure that this function can be computed only by large circuits, which implies that the refutation has to be large.

The interpolation method has been used to obtain lower bounds for several other proof systems, including Resolution. This is our fourth and last example the strong and fruitful connections between computational complexity and proof complexity. To summarize, here are the four examples we saw:

1. We showed that a polynomially bounded proof system exists if and only if NP is closed under complement.

2. We designed a polynomial-size Sequent Calculus proof of the pigeonhole principle using an NC^1 iterated addition circuit.

3. We proved a lower bound on decision trees to prove a lower bound for Tree Resolution.
4. We used a circuit complexity lower bound to prove a lower bound for Cutting Planes.