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THE VISUALLY DISTINCT CONFIGURATIONS OF k SETS

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Two sets of sets, C_0 and C_1, are said to be visually equivalent if there is a 1–1 mapping m from C_0 onto C_1 such that for every $S, T \in C_0$, $S \cap T = \emptyset$ if and only if $m(S) \cap m(T) = \emptyset$ and $S \subseteq T$ if and only if $m(S) \subseteq m(T)$. We find estimates for $V(k)$, the number of equivalence classes of this relation on sets of k sets, for finite and infinite k. Our main results are that for finite k, $\frac{1}{2}k^2 - k \log k < \log V(k) < \alpha k^2 + \beta k + \log k$, where α and β are approximately 0.7255 and 2.5323 respectively, and there is a set N of cardinality $\frac{1}{2}(k^2 + k)$ such that there are $V(k)$ visually distinct sets of k subsets of N.

1. Introduction

We consider the following equivalence relation on sets of sets. Let C_0 and C_1 be two sets whose members are sets. We permit C_0, C_1 and their members to be infinite. Then C_0 and C_1 are visually equivalent (written $C_0 \equiv C_1$) if and only if there is a 1–1 mapping from C_0 onto C_1 such that for all $S, T \in C_0$

$$S \cap T = \emptyset \quad \text{if and only if} \quad m(S) \cap m(T) = \emptyset,$$

and

$$S \subseteq T \quad \text{if and only if} \quad m(S) \subseteq m(T).$$

This definition is due to S.M. Ulam, who proposed it as a characterization of the way in which configurations of sets are often viewed. For instance, if we consider a configuration consisting of unlabelled subsets of a space, say the plane, we may not be interested in the individual points in the subsets; we may only want to know how the subsets overlap or contain each other.

There is, of course, a similar equivalence relation if we distinguish the sets in C_0 and C_1, say $C_i = \{S_{ia} : a \in K\}$ for $i = 0, 1$, where K is a set. We put $C_0 \sim C_1$ if and only if for every $a, b \in K$,

$$S_{0a} \cap S_{0b} = \emptyset \quad \text{if and only if} \quad S_{1a} \cap S_{1b} = \emptyset,$$

and

$$S_{0a} \subseteq S_{0b} \quad \text{if and only if} \quad S_{1a} \subseteq S_{1b}.$$

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Let $U(k)$ and $V(k)$ be the number of equivalence classes of $\sim$ and $\equiv$ respectively on sets of k sets. Our main results pertain to finite k. We show that

$$\tfrac{1}{2}(k^2-k)\leqslant \log U(k)<\alpha k^2+\beta k+\log k,$$

where α and β are approximately 0.7255 and 2.5323 respectively and log is $\log_2$. It follows immediately that

$$\tfrac{1}{2}k^2-k\log k<\log V(k)<\alpha k^2+\beta k+\log k.$$

We also investigate the number of visually distinct configurations of k sets when the sets in question are subsets of an n-element set. Here our results are weaker, but we show that if k is finite and $n\geqslant\tfrac{1}{2}(k^2+k)$ then there is an n-element set N such that each equivalence class of $\sim$ (and therefore $\equiv$) on sets of k sets is represented by some set of subsets of N, and if $n\geqslant k$, then there is an n-element set N such that more than $2^{k^2/4+3k/2-3\log k}$ classes of $\sim$ are represented by sets of k subsets of N.

Some of our methods easily extend to infinite k. We show that $V(k)\leqslant U(k)\leqslant 2^{2k}$, or (assuming the axiom of choice) $V(k)=U(k)=2^k$. Also, if $n\geqslant 2^k$ (or $n\geqslant k$ and we assume the axiom of choice) then there is an n-element set N such that each equivalence class of $\sim$ and $\equiv$ on sets of k sets is represented by some set of subsets of N.

Before proving our results, we give the values of $U(k)$ and $V(k)$ for $k\leqslant 4$:

k	$U(k)$	$V(k)$
1	1	1
2	4	3
3	43	11
4	928	63

2. First results

In what follows, K will be a set. Let $k=|K|$, the cardinality of K, and $K^2=\{(a,b):a,b\in K\}$. For $G\subseteq K^2$ and $a,b\in K$, we put $G(a,b)$ if $(a,b)\in G$. $W(K)$ is the set of structures $\langle K,G,P\rangle$ satisfying the following three axioms:

(2.1) G is a reflexive and symmetric graph, i.e. $G\subseteq K^2$ and for all $a,b\in K$, $G(a,a)$, and $G(a,b)\leftrightarrow G(b,a)$.

(2.2) P is a partial order, i.e. $P\subseteq K^2$ and for all $a,b,c\in K$, $P(a,a)$, $P(a,b)$ and $P(b,a)\rightarrow a=b$, and $P(a,b)$ and $P(b,c)\rightarrow P(a,c)$.

(2.3) For all $a,b,c,d\in K$, $G(a,b)$ and $P(a,c)$ and $P(b,d)\rightarrow G(c,d)$.

We put $W(k)$ for $|W(K)|$ and $W(k)/\cong$ for the number of isomorphism classes of $W(K)$. Let $C(K)$ be the class of sets of the form $\{S_a:a\in K\}$, where each S_a is a set. We shall use the following mapping m from $W(K)$ into $C(K)$:

$$m(\langle K, G, P\rangle) = \{S_a : a \in K\},$$

where

$$S_a = \{\{b, c\} \subseteq K : P(b, a) \text{ or } P(c, a), \text{ and } G(b, c)\}.$$

Lemma 2.4. *For all $a, b \in K$,*

(i) *$G(a, b)$ if and only if $S_a \cap S_b \neq \emptyset$,*

(ii) *$P(a, b)$ if and only if $S_a \subseteq S_b$,*

(iii) *$a = b$ if and only if $S_a = S_b$.*

Proof. (i) Assume $G(a, b)$. Then since $P(a, a)$ and $P(b, b)$ by (2.2), $\{a, b\} \in S_a \cap S_b$.

Assume $\{c, d\} \in S_a \cap S_b$ for some $c, d \in K$. Then $G(c, d)$ and one of the following four cases holds:

(1) $P(c, a)$ and $P(c, b)$,

(2) $P(c, a)$ and $P(d, b)$,

(3) $P(d, a)$ and $P(c, b)$,

(4) $P(d, a)$ and $P(d, b)$.

Also, $G(c, c)$ and $G(d, d)$ by (2.1). Assuming (1), $G(a, b)$ follows from $G(c, c)$ and (2.3). Assuming (2), $G(a, b)$ follows from $G(c, d)$ and (2.3). The other cases are similar.

(ii) Assume $P(a, b)$, and take any $\{c, d\} \in S_a$. Then $G(c, d)$, and $P(c, a)$ or $P(d, a)$, say $P(c, a)$. Since $P(a, b)$, we have $P(c, b)$ by (2.2), and therefore $\{c, d\} \in S_b$, implying $S_a \subseteq S_b$.

Assume $S_a \subseteq S_b$. Then since $\{a\} \in S_a$ by (2.1) and (2.2), $\{a\} \in S_b$. Therefore $P(a, b)$.

(iii) follows from (2.2) and (ii) above.

Corollary 2.5. *For every $C \in C(K)$, there is some $D \in W(K)$ such that $m(D) \sim C$.*

Proof. Let $C = \{S_a : a \in K\}$. Then we take $D = \langle K, G, P\rangle$ where

$$G = \{(a, b) \in K^2 : S_a \cap S_b \neq \emptyset\} \quad \text{and} \quad P = \{(a, b) \in K^2 : S_a \subseteq S_b\}.$$

Clearly $D \in W(K)$, and $m(D) \sim C$ by Lemma 2.4.

Theorem 2.6. *There is a set N of cardinality $\leq 2^k$ such that every equivalence class of $\sim$ (and therefore $\equiv$) on sets of k sets is represented by some set of subsets of N. If k is infinite and we assume the axiom of choice, then $|N| = k$. If k is finite, then $|N| = \frac{1}{2}(k^2 + k)$.*

Proof. Let $N = \{\{a, b\} : a, b \in K\}$. Then m maps each $D \in W(K)$ to a set of subsets of N. The result then follows from Corollary 2.5.

Theorem 2.7. *$U(k)=W(k)$, and $V(k)=W(k)/\cong$.*

Proof. By Lemma 2.4, for every $D_0, D_1 \in W(K)$, $D_0 = D_1$ if and only if $m(D_0) \sim m(D_1)$. This, together with Corollary 2.5, implies the theorem.

Corollary 2.8. *For infinite k, $V(k) \leqslant U(k) \leqslant 2^{2k^2}$. Assuming the axiom of choice, $V(k)=U(k)=2^k$.*

Proof. By Theorem 2.7, $W(k)/\cong\ = V(k) \leqslant U(k)=W(k)$, and $W(k) \leqslant 2^{2k^2}$ by elementary cardinal arithmetic.

To complete the proof, we need only show that the axiom of choice implies the existence of 2^k nonisomorphic structures in $W(K)$. This is obvious because there are 2^k nonisomorphic graphs G which are reflexive and symmetric, and for each such G, $\langle K, G, P_0\rangle \in W(K)$, where $P_0=\{(a,a): a \in K\}$.

3. Inequalities

In this section we prove the inequalities for $U(k)$ and $V(k)$ for finite k. We remark that the following much shorter proof shows

$$\tfrac{1}{2}(k^2-k) \leqslant \log U(k) < 3k^2/4 + k + \log k.$$

Let $M(k)=|\{G \subseteq K^2 : G \text{ is reflexive and symmetric}\}|$ and $N(k)=|\{P \subseteq K^2 : P \text{ is a partial order}\}|$. Then, by Theorem 2.7,

$$\log M(k) \leqslant \log U(k) \leqslant \log M(k) + \log N(k).$$

Now $\log M(k)=\frac{1}{2}(k^2-k)$ is obvious, and $\log N(k) < k^2/4 + 3k/2 + \log k$ by a result of Kleitman and Rothschild [2]. The result follows immediately.

Most of this section is devoted to improving the upper bound for $\log U(k)$ slightly. We shall use sets $L \subseteq K^2$ such that

(3.1) (i) if $(a,b) \in L$ then $a \neq b$,

(ii) if $(a,b), (c,d) \in L$ then either $(a,b)=(c,d)$ or $\{a,b\} \cap \{c,d\} = \emptyset$.

If L is such a set, then $|L|=kx$, where $0 \leqslant x \leqslant \frac{1}{2}$. Letting x be any real number such that $0 \leqslant x \leqslant \frac{1}{2}$, let $A(K,x)$ be those $P \subseteq K^2$ such that there is no $L \subseteq P$ satisfying (3.1) above and $|L| \geqslant kx$.

Lemma 3.2. *$\log |\{\langle K, G, P\rangle \in W(K) : P \in A(K,x)\}| < k^2 p(x) + kq(x)$, where*

$$p(x) = -4x^2 + 4x + \tfrac{1}{2} \quad \textit{and} \quad q(x) = -2x + \tfrac{1}{2}.$$

Proof. For a given $P \in A(K,x)$, let L be a maximal subset of P satisfying (3.1), and let $M=\{a : \text{there is some } b \text{ such that } (a,b) \in L \text{ or } (b,a) \in L\}$. For every $(a,b) \in P$ such that $a \neq b$, either $a \in M$ or $b \in M$ by the maximality of L.

Now every $P \in A(K,x)$ can be obtained by choosing a set $M \subseteq K$ such that

$|M|<2kx$, and then choosing P such that for every $(a,b)\in P$, if $a\neq b$ then $a\in M$ or $b\in M$. Therefore

$$\log|A(K,x)|<k+k(k-1)-(k-2kx)(k-2kx-1).$$

Since there are $2^{(k^2-k)/2}$ reflexive, symmetric graphs on K, the lemma follows.

Lemma 3.3. $\log|\{\langle K,G,P\rangle\in W(K):P\notin A(K,x)\}|<k^2r(x)+ks(x)+\log k$, *where*

$$r(x)=(2+\tfrac{1}{2}\log 6-2\log 3)x^2+(\log 3-2)x+\tfrac{3}{4}$$

and

$$s(x)=(1+\tfrac{1}{2}\log 6-2\log 3)x+1+\log 3.$$

Proof. By [2] the logarithm of the number of partial orders on K is less than $k^2/4+3k/2+\log k$. For a given $P\notin A(K,x)$, let $L\subseteq P$ satisfy (3.1) and $kx\leqslant|L|\leqslant kx+1$, and let M be as in the proof of Lemma 3.2. Then K^2 is the disjoint union of M^2, $(M\times(K-M)\cup(K-M)\times M)$ and $(K-M)^2$, and having chosen P, the number of ways of choosing G is not more than the product of the number of ways of choosing subsets of these three sets subject to the constraints of axioms (2.1) and (2.3).

The number of ways of choosing $G\cap M^2$ is not more than $6^{(kx+1)kx/2}$. To see this, for each pair of directed edges $(a,b),(c,d)\in L$, there are at most 6 subsets of $\{a,b,c,d\}^2$ that satisfy (2.1) and (2.3). The lines in the diagrams of Fig. 1

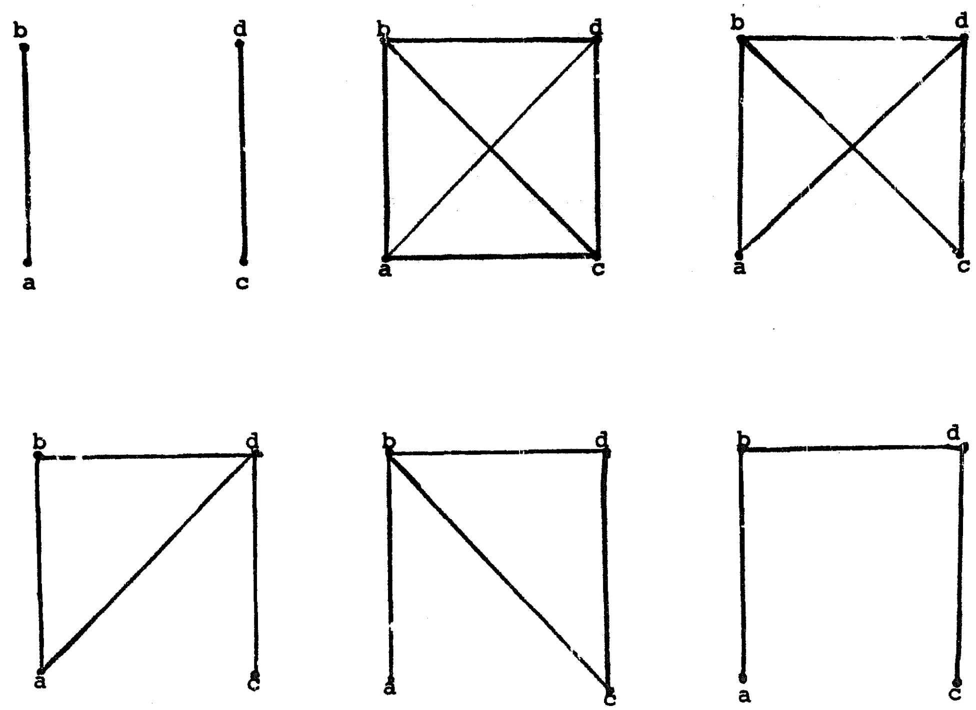

Fig. 1.

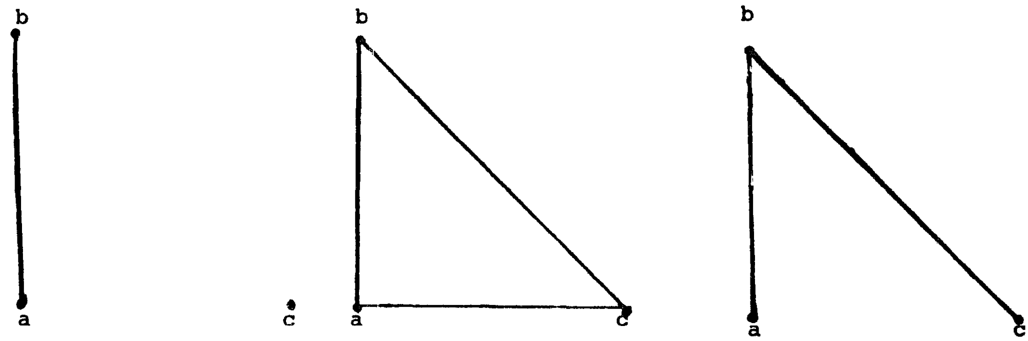

Fig. 2

represent the configurations of the undirected edges that satisfy (2.1) and (2.3), given that (a, b), $(c, d) \in P$. Since every element of $G \cap M^2$ occurs as an undirected edge in some such $\{a, b, c, d\}$, our claim follows.

The number of ways of choosing $G \cap (M \times (K-M) \cup (K-M) \times M)$ is not more than $3^{(kx+1)(k-2kx)}$. To see this, let $(a, b) \in L$ and $c \notin M$. There are at most 3 subsets of $\{a, b\} \times \{c\} \cup \{c\} \times \{a, b\}$ that satisfy (2.1) and (2.3) (see Fig. 2).

Lastly, there are at most $2^{(k-2kx)(k-2kx-1)/2}$ possibilities for $G \cap (K-M)^2$. The lemma then follows.

Theorem 3.4. $\frac{1}{2}(k^2-k) \leq \log U(k) < \alpha k^2 + \beta k + \log k$, *where* α *and* β *are approximately* 0.725545186 *and* 2.53232954 *respectively.*

Proof. As noted at the beginning of this section, the lower bound is trivial. To prove the upper bound, let p, q, r, and s be as in the previous two lemmas. Then, for fixed x such that $0 \leq x \leq \frac{1}{2}$,

$$U(k) = W(k) < 2^{k^2 p(x) + kq(x)} + 2^{k^2 r(x) + ks(x) + \log k}.$$

Letting $t(x) = \max(p(x), r(x))$ and $u(x) = \max(kq(x), ks(x) + \log k)$, $\log U(k) < k^2 t(x) + u(x)$.

Since p is increasing over $0 \leq x \leq \frac{1}{2}$ and r is decreasing, the minimum of t occurs where $p(x) = r(x)$. Solving this quadratic equation for x (we get $x = 0.059984428$) and letting $\alpha = p(x)$ and $u(x) = \beta k + \log k$, the upper bound follows.

Corollary 3.5. $\frac{1}{2}k^2 - k \log k < \log V(k) < \alpha k^2 + \beta k + \log k$.

Proof. The proof is immediate from Theorems 2.7, 3.4, and Stirling's formula for $k!$.

We now return to the question of how many visually distinct configurations of k sets there are when the sets are subsets of an n-element set (see Theorem 2.6 above).

Theorem 3.6. *Letting $v=\frac{1}{4}k^2+\frac{3}{2}k-3\log k$, there are more than 2^v equivalence classes of $\sim$ and $2^v/k!$ equivalence classes of $\equiv$ represented by subsets of K.*

Proof. Let P be any partial order on K. Let $C=\{S_a : a\in K\}$ where each $S_a=\{b\in K : P(b,a)\}$. Then for all $a, b\in K$, $S_a\subseteq S_b$ if and only if $P(a,b)$. The theorem then follows from [2].

4.

An open problem is to improve the bounds of Theorem 3.4. We expect that the upper bound can be lowered significantly, perhaps to $k^2/2+o(k^2)$, because axiom (2.3) is a strong condition that severely restricts the kinds of graphs and partial orders that can coexist. This restrictive power of (2.3) was exploited in Theorem 3.4, but it seems likely that it could be exploited further.

Some other open problems are: does $\lim_{k\to\infty}(\log V(k))/k^2$ exist, is $V(k)$ asymptotic to $U(k)/k!$ (by analogy with [1, 3, 4]), and can Theorems 2.6 and 3.6 be improved, i.e. if the k sets are subsets of an n-element set and n is not very large compared to k, then how many distinct configurations are there? Also, can one study an equivalence relation similar to to $\equiv$ or $\sim$ on the hierarchy of sets of sets of sets and so forth, as was done for isomorphism in [3]?

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